Linear Algebra II

Spring 2025
28th April

Lecture 3

Recall (Lecture 2)

Theorem 15.3 Let V be fin. gen. and let $F: V \to W$ be a linear map. Then

$$\dim V = \dim(\ker(F)) + \dim(\operatorname{im}(F)).$$

Ex 51: $3 = 1 + 2$

Example 51 Consider the linear map

$$D: P_2 \to P_2$$

$f(x) = ax^2 + bx + c \qquad f \mapsto f' \qquad D(f)(x) = 2ax + b$

$$\ker(D) = P_0, \quad \operatorname{im}(D) = P_1$$

Notice: P_n behaves like $\mathbb{R}^{n+1}$ "P_n and $\mathbb{R}^{n+1}$ are isomorphic"

$$ax^2 + bx + c \leftrightsquigarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Above linear map can be described as a lin. map $\mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We will make this precise in the following:

Definition 15.4 i) (Recall) A fct. $f: X \to Y$ is <u>invertible</u> if there exists a fct. $g: Y \to X$ with $f \circ g = id_Y$ and $g \circ f = id_X$. f is invertible iff f is bijective.

(ii) An invertible lin. map $F: V \to W$ is called an <u>isomorphism</u>. (isos = equal, morphe = form/shape)

(iii) Two vector spaces V and W are called isomorphic (Notation $V \cong W$) if there exists an isomorphism $F: V \to W$.

(In example: $P_2 \cong \mathbb{R}^3$, $ax^2+bx+c \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$)

Theorem 15.5 i) A lin. map $F: V \to W$ is an isomorphism iff $\ker(F) = \{0_V\}$ and $im(F) = W$.

ii) Let $F: V \to W$ be an isomorphism an $(b_1, \ldots, b_n)$ a basis of V. Then $(F(b_1), \ldots, F(b_n))$ is a basis of W.

iii) Let V, W be fin. gen. and $V \cong W$ then $\dim(V) = \dim(W)$.

iv) Let V, W be fin. gen. and $\dim(V) = \dim(W)$. Then for a lin. map $F: V \to W$ the following statements are equivalent:

(a) F is an isomorphism
(b) $\ker(F) = \{0_V\}$
(c) $im(F) = W$

Proposition 15.6 Let V be fin. gen. with basis $B = (b_1, \dots, b_n)$, i.e. $\dim(V) = n$. Then the <u>coordinate map</u>

$$C_B : \mathbb{R}^n \longrightarrow V$$
$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \longmapsto \sum_{i=1}^{n} \lambda_i b_i$$

is an isomorphism. The inverse is given by $C_B^{-1}(x) = [u]_B$ for $u \in V$.

Example 53 $V = P_2$, $B = (\underset{b_1}{x+1}, \underset{b_2}{x^2-1}, \underset{b_3}{x+3})$ basis of V.

$$C_B : \mathbb{R}^3 \longrightarrow P_2$$
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto a\overset{b_1}{(x+1)} + b\overset{b_2}{(x^2-1)} + c\overset{b_3}{(x+3)}$$
$$= (b)x^2 + (a+c)x + (a+3c)$$

$$C_B\left(\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right) = -x^2 + 3x + 8$$

$\Rightarrow$ If $f(x) = -x^2 + 3x + 8$ then $[f]_B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

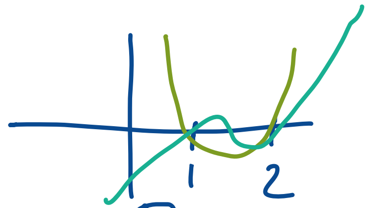
Corollary 15.7 Let V, W be fin. gen.. Then the following two statements are equivalent

i) $V \cong W$

ii) $\dim(V) = \dim(W)$.

(In particular: $V \cong \mathbb{R}^n$ for some $n \geq 1$)

Example 54 (Similar to HW 1)



$$U = \{ f \in P_3 \mid f(1) = f(2) = 0 \} \subset P_3$$

A possible basis of U is $B = (b_1, b_2)$

$$b_1(x) = (x-1)(x-2) = x^2 - 3x + 2 \quad , \quad b_2(x) = (x-1)(x-2)(x+3) = x^3 - 7x + 6$$

$\Longrightarrow \quad \dim U = 2 \Longrightarrow U \cong \mathbb{R}^2$

Isomorphism is given by
$C_B : \mathbb{R}^2 \to U$
$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \mapsto \lambda_1 b_1 + \lambda_2 b_2$
or
$C_B^{-1} : U \to \mathbb{R}^2$
$f \mapsto [f]_B$

Let $f(x) = (x-1)^2(x-2)$. What is $[f]_B$?

$$f(x) = \lambda_1 b_1(x) + \lambda_2 b_2(x) \quad \forall x \in \mathbb{R}.$$

$$(x-1)^2(x-2) = \lambda_1 (x-1)(x-2) + \lambda_2 (x-1)(x-2)(x+3)$$

$$x - 1 = \lambda_1 + \lambda_2 (x+3)$$

$$0 = \underbrace{(\lambda_2 - 1)}_{=0} x + \underbrace{\lambda_1 + 1 + 3\lambda_2}_{=0}$$

Here we use x and 1 are lin. indep.

$$\Longrightarrow \quad \lambda_2 = 1 \quad \lambda_1 = -4$$

$$[f]_B = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$$

§ 16 The matrix of a linear map

Goal: For fin. gen. vector spaces V, W and a linear map $F: V \to W$ define "the" matrix of F.

$$\begin{array}{ccc} V & \longrightarrow & W \\ \wr\wr & & \wr\wr \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

There are a lot of choices depending on bases of V and W.

Definition 16.1 Let V, W be fin. gen. vector spaces with bases $B_V = (v_1, \dots, v_n)$ and $B_W = (w_1, \dots, w_m)$. For a lin. map $F: V \to W$ we define <u>the matrix of F with respect to B_V and B_W</u> by

$$[F]_{B_V}^{B_W} := [C_{B_W}^{-1} \circ F \circ C_{B_V}].$$

$$\begin{tikzcd} V \arrow[r, "F"] & W \\ \mathbb{R}^n \arrow[u, "C_{B_V}"] \arrow[r, "G"'] & \mathbb{R}^m \arrow[u, "C_{B_W}"'] \end{tikzcd}$$

Here $\underbrace{C_{B_W}^{-1} \circ F}_{G} \circ C_{B_V}$ is the linear map $\mathbb{R}^n \to \mathbb{R}^m$ for which we defined the matrix $[..]$ in LA I. We have:

$$[G] = \begin{pmatrix} | & & | \\ G(e_1) & \dots & G(e_n) \\ | & & | \end{pmatrix}$$

And by $G(e_j) = C_{B_W}^{-1}\Big(\underbrace{F\big(\underbrace{C_{B_V}(e_j)}_{v_j}\big)}_{F(v_j)}\Big)$ we get

$$\underbrace{\phantom{C_{B_W}^{-1}\big(F(C_{B_V}(e_j))\big)}}_{[F(v_j)]_{B_W}}$$

$$[F]_{B_V}^{B_W} = \begin{pmatrix} | & & | \\ [F(v_1)]_{B_W} & \cdots & [F(v_n)]_{B_W} \\ | & & | \end{pmatrix}$$

Special case: $V=W$ and $F=$ identity (id_V):

__Definition 16.2__ Let $B_1 = (v_1, \dots, v_n)$ and $B_2 = (u_1, \dots, u_n)$ be bases of V. The __change-of-basis matrix from B_1 to B_2__ is the matrix

$$S_{B_1}^{B_2} = [\mathrm{id}_V]_{B_1}^{B_2} = [C_{B_2}^{-1} \circ C_{B_1}] = \begin{pmatrix} | & & | \\ [v_1]_{B_2} & \cdots & [v_n]_{B_2} \\ | & & | \end{pmatrix}.$$

For $v \in V$ we have

$$S_{B_1}^{B_2} [v]_{B_1} = [v]_{B_2}.$$

Example 55 Consider the linear map

$$D : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$$
$$f \longmapsto f'$$

$$\mathcal{B} = (\underset{b_1}{1}, \underset{b_2}{x}, \underset{b_3}{x^2}) \quad , \quad \mathcal{C} = (\underset{c_1}{2}, \underset{c_2}{2x+4}, \underset{c_3}{x^2})$$

We could calculate $\underline{[D]^{\mathcal{B}}_{\mathcal{B}}}$, $\underline{[D]^{\mathcal{C}}_{\mathcal{B}}}$, $\underline{[D]^{\mathcal{C}}_{\mathcal{C}}}$, $\underline{[D]^{\mathcal{B}}_{\mathcal{C}}}$

$$f(x) = ax^2+bx+c \xmapsto{\quad D \quad} \overset{D(f)(x)}{\overset{\parallel}{f'(x)}} = 2ax+b$$

$$\begin{tikzcd}
{[f]_{\mathcal{B}} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}} \arrow[rr, "{[D]^{\mathcal{B}}_{\mathcal{B}}}"] \arrow[rrd, "{[D]^{\mathcal{C}}_{\mathcal{B}}}"'] \arrow[d, "S^{\mathcal{C}}_{\mathcal{B}}"'] & & {[D(f)]_{\mathcal{B}} = \begin{pmatrix} b \\ 2a \\ 0 \end{pmatrix}} \\
{[f]_{\mathcal{C}} = \begin{pmatrix} \frac{1}{2}c-b \\ \frac{1}{2}b \\ a \end{pmatrix}} \arrow[rru, "{[D]^{\mathcal{B}}_{\mathcal{C}}}"] \arrow[rr, "{[D]^{\mathcal{C}}_{\mathcal{C}}}"'] & & {[D(f)]_{\mathcal{C}} = \begin{pmatrix} \frac{1}{2}b-2a \\ a \\ 0 \end{pmatrix}}
\end{tikzcd}$$

Calculate the matrices:

$$[D]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} | & | & | \\ [D(b_1)]_{\mathcal{B}} & [D(b_2)]_{\mathcal{B}} & [D(b_3)]_{\mathcal{B}} \\ | & | & | \end{pmatrix} = \boxed{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$[D]^C_B = \begin{pmatrix} \vert & \vert & \vert \\ [D(b_1)]_C & [D(b_2)]_C & [D(b_3)]_C \\ \vert & \vert & \vert \end{pmatrix} = \boxed{\begin{pmatrix} 0 & \frac{1}{2} & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$B = (\underset{b_1}{1}, \underset{b_2}{x}, \underset{b_3}{x^2}) \quad , \quad C = (\underset{c_1}{2}, \underset{c_2}{2x+4}, \underset{c_3}{x^2})$$

$$D(b_1) = 0 \;, \quad [D(b_1)]_B = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad [D(b_1)]_C = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D(b_2) = 1 \;, \quad [D(b_2)]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad [D(b_2)]_C = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

$$D(b_3) = 2x \;, \quad [D(b_3)]_B = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad [D(b_3)]_C = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$S^C_B = \begin{pmatrix} \vert & \vert & \vert \\ [b_1]_C & [b_2]_C & [b_3]_C \\ \vert & \vert & \vert \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$c_1 = \tfrac{1}{2} b_1 \,, \quad c_2 = \tfrac{1}{2} b_2 - b_1 \,, \quad c_3 = b_3$$

$$S^B_C = \begin{pmatrix} \vert & \vert & \vert \\ [c_1]_B & [c_2]_B & [c_3]_B \\ \vert & \vert & \vert \end{pmatrix} = \begin{pmatrix} 2 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \underset{S^B_C}{\left(\begin{array}{ccc|ccc} 2 & 4 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)} \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$S^B_C = (S^C_B)^{-1}$$

Remarks: i) We have $\left(S_{B_1}^{B_2}\right)^{-1} = S_{B_2}^{B_1}$

ii) If we have base B_1, B_2, B_3, B_4 of V and a linear map $F: V \to V$, then

$$[F]_{B_1}^{B_4} = S_{B_3}^{B_4} [F]_{B_2}^{B_3} S_{B_1}^{B_2}$$

Often: $B_1 = B_4$, $B_2 = B_3$

$$[F]_{B_1}^{B_1} \qquad\qquad [F]_{B_2}$$

Notation $\to$!!

$$[F]_{B_1} = S_{B_2}^{B_1} \underbrace{[F]_{B_2}^{B_2}} S_{B_1}^{B_2}$$

$$= \left(S_{B_1}^{B_2}\right)^{-1} [F]_{B_2} S_{B_1}^{B_2} .$$

iii) $$[F]_{B_1}^{2} = \left(S_{B_1}^{B_2}\right)^{-1} [F]_{B_2} \cancel{S_{B_1}^{B_2} \left(S_{B_1}^{B_2}\right)^{-1}} [F]_{B_2} S_{B_1}^{B_2}$$

$$= \left(S_{B_1}^{B_2}\right)^{-1} [F]_{B_2}^{2} S_{B_1}^{B_2}$$

Motivation: In applications we want to calculate $[F]_{B_1}^n$. For this we try to find B_2 such that $[F]_{B_2}^n$ is easy to calculate.

e.g. diagonal matrix