# Linear Algebra II

## Lecture 2

Spring 2025
$21^{th}$ April

Last time: Def vector space $(V, +, \cdot)$

**Example 46** i) $V = \mathbb{R}^n$

ii)
$$\mathcal{P}_n \subset \mathcal{P} \subset \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \subset \mathcal{C}^n(\mathbb{R},\mathbb{R}) \subset \mathcal{C}^0(\mathbb{R},\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})$$

There are all vector spaces.

iii) $V = \mathcal{J}$ = set of all infinite sequences "$\mathbb{R}^\infty$"

$\ni (a_n)_{\geq 1} = (a_1, a_2, a_3, \dots)$ $\quad 0 = (0, 0, \dots)$

$a = (a_n)$, $b = (b_n)$, $\quad a+b = (a_n + b_n)$
$\lambda a = (\lambda a_n)$

iv) Matrices: $V = \mathbb{R}^{m \times n}$ $\quad n = 0 = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$

v) If $(V, +, \cdot)$ is a vector space and $f: V \to W$ is a bijective map, then you can define for

$u, v \in W$: $\quad u \oplus v = f(f^{-1}(u) + f^{-1}(v))$

$\lambda \in \mathbb{R}$ $\quad \lambda \odot u = f(\lambda \cdot f^{-1}(u))$ $\quad$ (HW 1, Ex. 1)

and obtain a vector space $(W, \oplus, \odot)$.

**Definition 14.3** Let $(V,+,\cdot)$ be a vector space. A subset $U \subset V$ is a <u>subspace</u> if

i) $0 \in U$

ii) $\forall u,v \in U: \ u+v \in U$

iii) $\forall u \in U, \lambda \in \mathbb{R}: \ \lambda u \in U$

**Example 47** i) The sets in Ex. 46 ii) are subspaces

$$\mathcal{P}_n \subset \mathcal{P} \subset \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \subset \mathcal{C}^n(\mathbb{R},\mathbb{R}) \subset \mathcal{C}^0(\mathbb{R},\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})$$

ii) $\mathcal{J}^0 = \{ (a_n)_{n\geq 1} \in \mathcal{J} \mid \lim_{n\to\infty} a_n \text{ exists} \} \subset \mathcal{J}$

is a subspace.

Clearly $\lim_{n\to\infty} 0 = 0$

Calculus: If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist then

$\lim_{n\to\infty} a_n + b_n$ exists.

iii) $GL_n(\mathbb{R}) = \{ A \in \mathbb{R}^{n\times n} \mid A^{-1} \text{ exists} \} \subset \mathbb{R}^{n\times n}$

is not a subspace! $\quad 0 \notin GL_n(\mathbb{R})$

**Proposition 14.4** If $U \subset V$ is a subspace, then $U$ is also a vector space with the operations inherited from $V$.

**Proof:** Clear.

**Definition 14.5.** Let $(V, +, \cdot)$ be a vector space and $v_1, \dots, v_n \in V$.

i) The **span** of $v_1, \dots, v_n$ is given by

$$\text{span}\{v_1, \dots, v_n\} = \{ \underbrace{\sum_{j=1}^{n} \lambda_i \cdot v_i \in V}_{\text{linear combination.}} \mid \lambda_1, \dots, \lambda_n \in \mathbb{R} \}$$

ii) The elements $v_1, \dots, v_n$ **span $V$** if $V = \text{span}\{v_1, \dots, v_n\}$.

iii) $V$ is **finitely generated** if there exist $v_1, \dots, v_n$ with $\text{span}\{v_1, \dots, v_n\} = V$.

NEW

(In LA I everything was fin. gen.!)

iv) The elements $v_1, \dots, v_n$ are **linearly independent** if

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \implies \lambda_1 = \dots = \lambda_n = 0$$

v) If $V = \text{span}\{v_1, \dots, v_n\}$ and $v_1, \dots, v_n$ are lin. indep., then $B = (v_1, \dots, v_n)$ is a **basis** of $V$.

**Example 48** i) Set $f_j(x) = x^j$. Then $f_0, f_1, \dots, f_n$ are linearly independent.

We will show this later by using determinants.

For $n=3$ (HW 1, Ex 3) please check it directly. For this use different values for $x$ (e.g. $-1, 0, 1, 2$) and use LAI.

In particular, $\mathcal{P}_n = \operatorname{span}\{f_0, \dots, f_n\}$ is fin. gen.

ii) $\mathcal{P}$ is <u>not</u> fin. gen., because if $g_1, \dots, g_n \in \mathcal{P}$ with $\mathcal{P} = \operatorname{span}\{g_1, \dots, g_n\}$ then set $d = \max_{i=1,\dots,n}\{\deg(g_i)\}$

degree of $g_i$

Then $f_{d+1}$ is not in $\operatorname{span}\{g_1, \dots, g_n\}$ by i).

Check yourself:

Prop. 14.6, Lemma 14.7, Thm 14.8 (existence of basis)

Def. 14.9 (dim), Cor 14.10,

Def 14.13: coordinates & coordinate vector.

# § 15 Linear maps

In the following we consider vector spaces $V, W$ and write $+_V, \cdot_V$ / $+_W, \cdot_W$ for addition/scalar multiplication and $0_V$ / $0_W$ for the neutral elements. Later we will usually just write "+", "·" and "0".

<u>Definition 15.1</u> Let $(V, +_V, \cdot_V)$, $(W, +_W, \cdot_W)$ be vector spaces.

A <u>linear map</u> is a function $F: V \to W$ satisfying

i) $F(u +_V v) = F(u) +_W F(v)$

ii) $F(\lambda \cdot_V u) = \lambda \cdot_W F(u)$

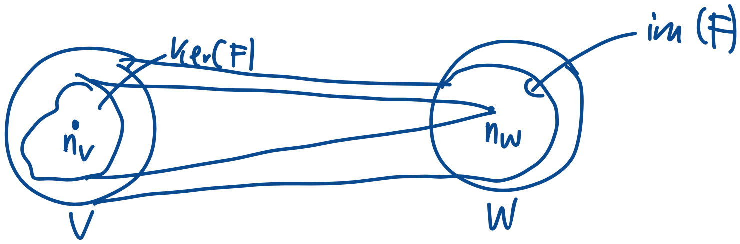
<u>Definition 15.2</u> Let $F: V \to W$ be a linear map.

i) The Kernel of $F$ is

$$\mathrm{Ker}(F) = \{ u \in V \mid F(u) = 0_W \} \subset V$$

ii) The image of $F$ is

$$\mathrm{im}(F) = \{ w \in V \mid \exists u \in V : w = F(u) \} \subset W$$



**Example 50** i) The map

$(n\geq 1)$: $D: \mathcal{C}^n(\mathbb{R},\mathbb{R}) \longrightarrow \mathcal{F}(\mathbb{R},\mathbb{R})$

$f \longmapsto f'$

is linear, since

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$ $\quad \forall f,g \in \mathcal{C}^n(\mathbb{R},\mathbb{R})$

$D(\lambda f) = \lambda D(f)$ $\quad \lambda \in \mathbb{R}$

$\ker(D) = \{\text{constant functions}\} = \mathcal{P}_0$

$\operatorname{im}(D) = \mathcal{C}^{n-1}(\mathbb{R},\mathbb{R})$

ii) The map

$ev_a: \mathcal{F}(\mathbb{R},\mathbb{R}) \longmapsto \mathbb{R}$

$f \longmapsto f(a)$

is linear for any $a \in \mathbb{R}$.

$ev_a(f+g) = (f+g)(a) = f(a) + g(a)$

$\ker(ev_a) = \{f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f(a) = 0\}$ $= ev_a(f) + ev_a(g).$

$\operatorname{im}(ev_a) = \mathbb{R}$

iii) $F: \mathbb{R}^{2\times 2} \to \mathbb{R}^2$ is linear
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ d \end{pmatrix}$

$\mathrm{Ker}(F) = \mathrm{span}\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$, $\dim(\mathrm{Ker}(F)) = 2$

$\mathrm{im}(F) = \mathbb{R}^2$ $\dim(\mathrm{im}(F)) = 2$

$\dim \mathbb{R}^{2\times 2} = 4 = 2+2$

<u>Theorem</u> 15.3 Let $V$ be fin. gen. and let $F: V \to W$ be a linear map. Then

$$\dim V = \underset{\overset{\shortparallel}{\ell}}{\dim(\mathrm{Ker}(F))} + \underset{\overset{\shortparallel}{n}}{\dim(\mathrm{im}(F))}.$$

Proof sketch: • $\mathrm{Ker}(F) \subset V$ subspace, i.e $\mathrm{Ker}(F)$ is fin. gen.

Let $(v_1, \dots, v_\ell)$ be a basis of $\mathrm{Ker}(F)$.

- We can extent this to a basis $(v_1, \dots, v_\ell, u_1, \dots, u_n)$ of $V$. (Thm. 1.8)
- Show: $(F(u_1), \dots, F(u_n))$ is a basis of $\mathrm{im}(F)$. (Think about that!)

**Example 51** Consider the linear map

$$D : \mathbb{P}_2 \longrightarrow \mathbb{P}_2$$
$$f \longmapsto f'$$

$f(x) = ax^2 + bx + c$ , $D(f)(x) = 2ax + b$

$\ker(D) = \mathbb{P}_0$ , $\operatorname{im}(D) = \mathbb{P}_1$

$\dim \mathbb{P}_n = n+1$ , $B = (1, x, \dots, x^n)$

will later show that this is a basis for arbitrary $n$ (Vandermonde determinant)

Notice: $\mathbb{P}_n$ behaves like $\mathbb{R}^{n+1}$ "$\mathbb{P}_n$ and $\mathbb{R}^{n+1}$ are isomorphic"

$$ax^2 + bx + c \quad \rightsquigarrow \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Above linear map can be described as a lin. map $\mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 2a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

We will make this precise in the following:

**Definition 15.4** i) (Recall) A fct. $f: X \to Y$ is <u>invertible</u> if there exists a fct. $g: Y \to X$ with $f \circ g = id_Y$ and $g \circ f = id_X$. $f$ is invertible iff $f$ is bijective.

ii) An invertible lin. map $F: V \to W$ is called an <u>isomorphism</u>. (isos = equal, morphe = form / shape)

iii) Two vector spaces $V$ and $W$ are called isomorphic (Notation $V \cong W$) if there exists an isomorphism $F: V \to W$.

(In example: $P_2 \cong \mathbb{R}^3$, $ax^2+bx+c \mapsto \begin{pmatrix} a \\ b \\ c \end{pmatrix}$)

**Theorem 15.5** i) A lin. map $F: V \to W$ is an isomorphism iff $Ker(F) = \{0_V\}$ and $im(F) = W$.

ii) Let $F: V \to W$ be an isomorphism an $(b_1, \dots, b_n)$ a basis of $V$. Then $(F(b_1), \dots, F(b_n))$ is a basis of $W$.

iii) Let $V, W$ be fin. gen. and $V \cong W$ then $\dim(V) = \dim(W)$.

iv) Let $V, W$ be fin. gen. and $\dim(V) = \dim(W)$. Then for a lin. map $F: V \to W$ the following statements are equivalent:

(a) $F$ is an isomorphism
(b) $Ker(F) = \{0_V\}$
(c) $im(F) = W$

**Proposition 15.6** Let $V$ be fin. gen. with basis $B = (b_1, \dots, b_n)$, i.e. $\dim(V) = n$. Then the <u>coordinate map</u>

$$C_B : \mathbb{R}^n \longrightarrow V$$
$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \longmapsto \sum_{i=1}^{n} \lambda_i b_i$$

is an isomorphism. The inverse is given by $C_B^{-1}(x) = [u]_B$ for $u \in V$.

**Example 53** $V = P_2$, $B = (\underset{b_1}{x+1}, \underset{b_2}{x^2-1}, \underset{b_3}{x+3})$ basis of $V$.

$$C_B : \mathbb{R}^3 \to P_2$$
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto a(\overset{b_1}{x+1}) + b(\overset{b_2}{x^2-1}) + c(\overset{b_3}{x+3})$$
$$= (b)x^2 + (a+c)x + (a+3c)$$

$$C_B\left(\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right) = -x^2 + 3x + 8$$

$\Rightarrow$ If $f(x) = -x^2 + 3x + 8$ then $[f]_B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

**Corollary 15.7** Let $V, W$ be fin. gen.. Then the following two statements are equivalent

i) $V \cong W$

ii) $\dim(V) = \dim(W)$.

(In particular: $V \cong \mathbb{R}^n$ for some $n \geq 1$)