Linear Algebra II

Spring 2025
14^{th} April

Lecture 1

(1– 13 LA I)

§ 14 Vector spaces

Vector space = A space having "addition" and "scalar multiplication" such that all "usual computation" rules as in $\mathbb{R}^n$ are satisfied.

Example 45 $\mathcal{F}(\mathbb{R},\mathbb{R})$: all functions $\mathbb{R} \to \mathbb{R}$

$e^x \in$ sin $\in$ cos $\in$ polynomials $\in$

If $f, g \in \mathcal{F}(\mathbb{R},\mathbb{R})$ we can define

$$(f+g)(x) := f(x) + g(x) \qquad \text{addition}$$

$\lambda \in \mathbb{R}$ $$(\lambda \cdot f)(x) := \lambda \cdot f(x) \qquad \text{scalar multiplication}$$

Consider the subset $U = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f'' = f \}$

1) The function $n(x) = 0 \ \forall x \in \mathbb{R}$ is in U. "zero function"

2) If $f, g \in U$ then $(f+g)''(x) = f''(x) + g''(x)$

$$= f(x) + g(x) = (f+g)(x)$$

$\Rightarrow f+g \in U$

3) If $f \in U$ and $\lambda \in \mathbb{R}$: $(\lambda f)''(x) = \lambda f''(x) = \lambda f(x)$

$\Rightarrow \lambda f \in U$

$\rightsquigarrow$ "U is a subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$"

Elements in U: $f(x) = e^x$, $g(x) = e^{-x}$

Also their linear combinations $\lambda_1 f + \lambda_2 g \in U$
$\lambda_1, \lambda_2 \in \mathbb{R}$

One can show that these are all, i.e. "$U = \text{span}\{f, g\}$".

Moreover if $\underbrace{\lambda_1 f + \lambda_2 g = n}_{\text{this means } \lambda_1 f(x) + \lambda_2 g(x) = n(x) = 0 \ \forall x \in \mathbb{R}}$ then $\lambda_1 = \lambda_2 = 0$

To show this consider different values of x

e.g. $x=0$ $\lambda_1 f_1(0) + \lambda_2 g(0)$

$= \lambda_1 + \lambda_2 = 0$

$x = 1$: $\lambda_1 e + \lambda_1 \frac{1}{e} = 0$

LAI $\Rightarrow \lambda_1 = \lambda_2 = 0$

$\Rightarrow f, g$ are "linearly independent"

$\Rightarrow$ "(f, g) is a basis of U"

$\Rightarrow$ "$\dim U = 2$" $\rightsquigarrow$ "$U \cong \mathbb{R}^2$"
"isomorphic"

Now let us make the notion precise:

<u>Definition 14.1</u> A <u>(real) vector space</u> is a tuple $(V,+,\cdot)$, where V is a set and $+$ and $\cdot$ are two functions

$$\overset{\text{addition}}{+}: V\times V \longrightarrow V \qquad\qquad \overset{\text{scalar multiplication}}{\cdot}: \mathbb{R}\times V \longrightarrow V$$
$$(u,v) \longmapsto u+v \quad , \qquad (\lambda,v) \longmapsto \lambda\cdot v = \lambda v$$

satisfying the following properties:

- Properties of addition

Example: $V=\mathbb{R}^n$

(A.1) $\forall u,v,w\in V$: $(u+v)+w = u+(v+w)$ (Associativity) ✓

(A.2) $\forall u,v\in V$: $u+v = v+u$ (Commutativity) ✓

$n=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$

(A.3) $\exists n\in V, \forall u\in V$: $n+u = u$ (Identity/neutral element of addition)

(A.4) $\forall u\in V, \exists v\in V$: $u+v = n$ (Inverse element of addition)

$u=\begin{pmatrix}u_1\\ \vdots\\ u_n\end{pmatrix}$, $v=\begin{pmatrix}-u_1\\ \vdots\\ -u_n\end{pmatrix} = (-1)\cdot u = -u$

- Compatibility of $+$ and $\cdot$:

(C.1) $\forall u,v\in V, \lambda\in\mathbb{R}$: $\lambda\cdot(u+v) = \lambda\cdot u + \lambda\cdot v$ (Distributivity I)

(C.2) $\forall u\in V, \lambda,\mu\in\mathbb{R}$: $(\lambda \underset{\text{in } \mathbb{R}}{+} \mu)\cdot u = \lambda\cdot u + \mu\cdot u$ (Distributivity II)

(C.3) $\forall u\in V, \lambda,\mu\in\mathbb{R}$: $\lambda\cdot(\mu\cdot u) = (\lambda\mu)\cdot u$

(C.4) $\forall u\in V$: $1\cdot u = u$

- If $+$ and $\cdot$ are clear from context we just write V instead of $(V,+,\cdot)$.
- "$+$" and "$\cdot$" are the usual symbols used, but any other symbol is possible. If we consider different vector spaces at the same time we might use other symbols, e.g. "$\oplus$", "$\odot$" or $(V, +_V, \cdot_V)$, $(W, +_W, \cdot_W)$

Proposition 14.2 Let $(V,+,\cdot)$ be a vector space and $u \in V$.

i) $u + n = u$

ii) If $n, \tilde{n} \in V$ both satisfy (A.3) then $n = \tilde{n}$
(The neutral element is unige. Notation $n = 0$)

If you see this symbol always make sure what it means!!

iii) If for a fixed $u \in V$ the elements $v, \tilde{v} \in V$ both satisfy (A.4), i.e. $u+v = u+\tilde{v} = n$, then $v = \tilde{v}$.
(The inverse is unique. Notation $v =: -u$)

iv) $-u = (-1)\cdot u$ (i.e. $u + (-1)u = n$)

Proof: Do yourself (good Exercise!) or see lecture notes.

Example 46 Example of vector spaces:

i) $V = \mathbb{R}^n$, $n = 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, $-u = \begin{pmatrix} -u_1 \\ \vdots \\ -u_n \end{pmatrix}$

ii) $V = \mathcal{F}(\mathbb{R}, \mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \}$ with addition & scalar mult. from Example 1.

$0 = n \leftarrow$ zero function $(n(x) = 0 \ \forall x \in \mathbb{R})$

$$\mathcal{C}^0(\mathbb{R},\mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f \text{ is continuous} \}$$
$$\cup$$
$$\mathcal{C}^n(\mathbb{R},\mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f^{(n)} \text{ exists and is continuous} \}$$
$$\cup$$
smooth functions
$$\mathcal{C}^\infty(\mathbb{R},\mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f^{(n)} \text{ exists for all } n \geq 0 \}$$
$$\cup$$
$$\mathcal{P} = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f \text{ is a polynomial fct.} \}$$
$$\cup$$
$$\mathcal{P}_n = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f(x) = \sum_{j=0}^{n} a_j x^j \text{ for some } a_1,\dots,a_n \in \mathbb{R} \}$$

These are all vector spaces.

iii) $V = \mathcal{J}$ = set of all infinite sequences "$\mathbb{R}^\infty$"

$\ni (a_n)_{\geq 1} = (a_1, a_2, a_3, \dots)$ $0 = (0, 0, \dots)$

$a = (a_n)$, $b = (b_n)$, $a + b = (a_n + b_n)$
$\lambda a = (\lambda a_n)$

iv) Matrices: $V = \mathbb{R}^{m \times n}$ $n = 0 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}$

v) If $(V, +, \cdot)$ is a vector space and $f: V \to W$ is a bijective map, then you can define for

$u, v \in W$: $u \oplus v = f(f^{-1}(u) + f^{-1}(v))$

$\lambda \in \mathbb{R}$ $\lambda \odot u = f(\lambda \cdot f^{-1}(u))$ (HW 1, Ex. 1)

and obtain a vector space $(W, \oplus, \odot)$.

Definition 14.3 Let $(V,+,\cdot)$ be a vector space. A subset $U \subset V$ is a *subspace* if

i) $0 \in U$

ii) $\forall u, v \in U: \; u+v \in U$

iii) $\forall u \in U, \lambda \in \mathbb{R}: \; \lambda u \in U$

Example 47 i) The sets in Ex. 46 ii) are subspaces

$$\mathcal{P}_n \subset \mathcal{P} \subset \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \subset \mathcal{C}^n(\mathbb{R},\mathbb{R}) \subset \mathcal{C}^0(\mathbb{R},\mathbb{R}) \subset \mathcal{F}(\mathbb{R},\mathbb{R})$$

ii) $$\mathcal{J}^0 = \{ (a_n)_{n \geq 1} \in \mathcal{J} \mid \lim_{n\to\infty} a_n \text{ exists} \} \subset \mathcal{J}$$

is a subspace.

Clearly $\lim_{n\to\infty} 0 = 0$

Calculus: If $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist then

$\lim_{n\to\infty} a_n + b_n$ exists.

iii) $$GL_n(\mathbb{R}) = \{ A \in \mathbb{R}^{n\times n} \mid A^{-1} \text{ exists} \} \subset \mathbb{R}^{n \times n}$$

is not a subspace! $\quad 0 \notin GL_n(\mathbb{R})$

Proposition 14.4 If $U \subset V$ is a subspace, then U is also a vector space with the operations inherited from V.

Proof: Clear.

Definition 14.5. Let $(V,+,\cdot)$ be a vector space and $v_1,\dots,v_n \in V$.

i) The *span* of $v_1,\dots,v_n$ is given by

$$\operatorname{span}\{v_1,\dots,v_n\} = \Big\{ \underbrace{\sum_{j=1}^{n} \lambda_i \cdot v_i}_{\text{linear combination.}} \in V \;\Big|\; \lambda_1,\dots,\lambda_n \in \mathbb{R} \Big\}$$

ii) The elements $v_1,\dots,v_n$ *span V* if $V = \operatorname{span}\{v_1,\dots,v_n\}$.

iii) V is *finitely generated* if there exist $v_1,\dots,v_n$ with $\operatorname{span}\{v_1,\dots,v_n\} = V$.
NEW
(In LA I everything was fin. gen.!)

iv) The elements $v_1,\dots,v_n$ are *linearly independent* if

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \Rightarrow \lambda_1 = \dots = \lambda_n = 0$$

v) If $V = \operatorname{span}\{v_1,\dots,v_n\}$ and $v_1,\dots,v_n$ are lin. indep., then $B = (v_1,\dots,v_n)$ is a *basis* of V.

Example 48 i) Set $f_j(x) = x^j$. Then $f_0, f_1, \dots, f_n$ are linearly independent.

We will show this later by using determinants.

For $n=3$ (HW 1, Ex 2) please check it directly. For this use different values for x (e.g. $-1, 0, 1, 2$) and use LA I.

In particular, $\mathcal{P}_n = \mathrm{span}\{f_0, \dots, f_n\}$ is fin. gen.

ii) $\mathcal{P}$ is <u>not</u> fin. gen., because if $g_1, \dots, g_n \in \mathcal{P}$ with $\mathcal{P} = \mathrm{span}\{g_1, \dots, g_n\}$ then set $d = \max\limits_{i=1,\dots,n}\{\deg(g_i)\}$

degree of g_i

Then f_{d+1} is not in $\mathrm{span}\{g_1, \dots, g_n\}$ by i).

Homework 1, Exercise 0: Read the rest of chapter 14.