Homework 2: Linear maps and their matrices

Deadline: 15th May (23:55 JST), 2025

Exercise 1. (2+2+2+2+2 = 10 Points) For $n \geq 0$ we define the map $H_n : \mathcal{P}_n \to \mathbb{R}^3$ for a $p \in \mathcal{P}_n$ by

$$H_n(p) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}.$$

(i) Show that H_n is a linear map for any $n \geq 0$.

(ii) Show that H_2 is an isomorphism and calculate the inverse of H_2.

(iii) Determine $[H_4]_B^C$, where $B = (1, x-1, x^2+1, x^3-1, x^4+1)$ and C is the standard basis of $\mathbb{R}^3$.

(iv) Check if H_1 and H_3 are injective and/or surjective.

(v) Determine a basis of $\operatorname{im}(H_1)$ and $\ker(H_3)$.

Exercise 2. (1+1+2+2 = 6 Points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}. \qquad (n \geq 2)$$

In this exercise, we want to prove the following explicit formula

$$F_n = \frac{1}{2^n\sqrt{5}}\left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right). \qquad (🐱)$$

For this follow the following steps:

(i) Find a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$, such that $F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for $n \geq 1$, where $F^n = \underbrace{F \circ \cdots \circ F}_{n}$.

(ii) Define the following two bases of $\mathbb{R}^2$ (you do not need to check that these are bases):

$$B_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right), \quad B_2 = \left(\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}\right).$$

Determine the change-of-basis matrices $S_{B_1}^{B_2}$ and $\left(S_{B_1}^{B_2}\right)^{-1}$.

(iii) Calculate $[F]_{B_1}$ and $[F]_{B_2}$.

(iv) Calculate $[F]_{B_1}^n$ by using

$$[F]_{B_1} = \left(S_{B_1}^{B_2}\right)^{-1} [F]_{B_2} S_{B_1}^{B_2}$$

and prove (🐱) by using (i).

Exercise 3. (2+2+2 = 6 Points) We define the space of Fibonacci sequences by

$$\mathcal{F} = \{(a_n)_{n\geq 0} \in \mathcal{J} \mid a_n = a_{n-1} + a_{n-2} \text{ for all } n \geq 2\},$$

where $\mathcal{J}$ denotes the vector space of all infinite sequences (see Lecture 1).

(i) Show that $\mathcal{F}$ is a vector space with the addition and scalar multiplication coming from $\mathcal{J}$.

(ii) Show that $\mathcal{F}$ is finitely generated and find a basis B of $\mathcal{F}$.

(iii) Define the map $G : \mathcal{F} \to \mathcal{F}$ on a sequence $a = (a_n)_{n\geq 0}$ by $G(a) = b$, where $b = (b_n)_{\geq 0}$ is given by $b_n = a_{n+2}$. Show that G is an isomorphism and determine $[G]_B$, where B is the basis in (ii).