Linear Algebra II & Mathematics Tutorial 2b Instructor: Henrik Bachmann
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Tutorial 5
9th May

Homework 2: Linear maps and their matrices

Deadline: 20th May (23:55 JST), 2024

Exercise 1. (2+2+2+2+2 = 10 Points) For $n \geq 0$ we define the map $H_n : \mathcal{P}_n \to \mathbb{R}^3$ for a $p \in \mathcal{P}_n$ by

$$H_n(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}.$$

(i) Show that H_n is a linear map for any $n \geq 0$.

(ii) Show that H_2 is an isomorphism and calculate the inverse of H_2.

(iii) Determine $[H_4]^C_B$, where $B = (1, x-1, x^2+1, x^3-1, x^4+1)$ and C is the standard basis of $\mathbb{R}^3$.

(iv) Check if H_1 and H_3 are injective and/or surjective.

(v) Determine a basis of $\operatorname{im}(H_1)$ and $\ker(H_3)$.

Exercise 2. (1+1+2+2 = 6 Points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}. \qquad (n \geq 2)$$

In this exercise we want to prove the following explicit formula

$$F_n = \frac{1}{2^n \sqrt{5}} \left((1+\sqrt{5})^n - (1-\sqrt{5})^n \right). \qquad (😸)$$

For this follow the following steps:

(i) Find a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$, such that $F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for $n \geq 1$, where $F^n = \underbrace{F \circ \cdots \circ F}_{n}$.

(ii) We define the following two bases of $\mathbb{R}^2$:

$$B_1 = \left(\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{b_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{b_2} \right), \quad B_2 = \left(\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} \right).$$

Determine the change-of-basis matrices $S^{B_2}_{B_1}$ and $\left(S^{B_2}_{B_1}\right)^{-1}$.

$$S^{B_2}_{B_1} = \begin{pmatrix} | & | \\ [b_1]_{B_2} & [b_2]_{B_2} \\ | & | \end{pmatrix}$$

(iii) Calculate $[F]_{B_1}$ and $[F]_{B_2}$.

(iv) Calculate $[F]^n_{B_1}$ by using

$$[F]_{B_1} = \left(S^{B_2}_{B_1}\right)^{-1} [F]_{B_2} S^{B_2}_{B_1}$$

and prove $(\boxed{😸})$ by using (i).

Exercise 3. (2+2+2 = 6 Points) We define the space of Fibonacci sequences by

$$\mathcal{F} = \{(a_n)_{n\geq 0} \in \mathcal{J} \mid a_n = a_{n-1} + a_{n-2} \text{ for all } n \geq 2\},$$

where $\mathcal{J}$ denotes the vector space of all infinite sequences (see Lecture 1).

(i) Show that $\mathcal{F}$ is a vector space with the addition and scalar multiplication coming from $\mathcal{J}$.

(ii) Show that $\mathcal{F}$ is finitely generated and find a basis B of $\mathcal{F}$.

(iii) Define the map $G : \mathcal{F} \to \mathcal{F}$ on a sequence $a = (a_n)_{n\geq 0}$ by $G(a) = b$, where $b = (b_n)_{\geq 0}$ is given by $b_n = a_{n+2}$. Show that G is an isomorphism and determine $[G]_B$, where B is the basis in (ii).

For:

Ex 2 Let $F: \mathbb{R}^2 \to \mathbb{R}^2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} 2x_1 \\ -x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$F\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$F^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = F\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2-3 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$F^3\begin{pmatrix} 1 \\ 0 \end{pmatrix} = F\begin{pmatrix} 4 \\ -5 \end{pmatrix} = \begin{pmatrix} 8 \\ -4-3\cdot 5 \end{pmatrix} = \begin{pmatrix} 8 \\ -19 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Goal: Calculate $F^n\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for $n \geq 1$

$$\Vert$$

$$\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Notice: $\begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix} = [F] = [F]_B \quad B = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

Idea: Find basis C of $\mathbb{R}^2$ such that

$[F]_C^n$ is easy to calculate.

$[F]_C = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \rightsquigarrow [F]_C^n = \begin{pmatrix} d_1^n & 0 \\ 0 & d_2^n \end{pmatrix}$

Motivation: $[F]_B = S_C^B [F]_C S_B^C$

Then $[F]_B^2 = S_C^B [F]_C \underbrace{S_B^C S_C^B}_{I_2} [F]_C S_B^C$

$$= S_C^B [F]_C^2 S_B^C$$

In general: $\underset{\text{want}}{[F]_B^n} = \underset{\text{easy}}{S_C^B [F]_C^n S_B^C}$.

In example: $C = \left(\underset{c_1}{\begin{pmatrix}1\\1\end{pmatrix}}, \underset{c_2}{\begin{pmatrix}0\\1\end{pmatrix}} \right)$

(Later we learn how to find such C)

Then $[F]_C = \begin{pmatrix} | & | \\ [F(c_1)]_C & [F(c_2)]_C \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

$F\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} = 2\begin{pmatrix}1\\1\end{pmatrix}$

$F\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\3\end{pmatrix} = 3\begin{pmatrix}0\\1\end{pmatrix}$

$$S_C^B = \begin{pmatrix} | & | \\ [c_1]_B & [c_2]_B \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$$S_B^C = \left(S_C^B\right)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\begin{array}{c} -1 \\ \downarrow \end{array} \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\Rightarrow \quad [F]_B^n = S_C^B [F]_C^n S_B^C.$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ -3^n & 3^n \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ 2^n - 3^n & 3^n \end{pmatrix}$$

$$\Rightarrow \quad [F]_B^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2^n \\ 2^n - 3^n \end{pmatrix}}}.$$