Tutorial 2: Vector spaces

A (real) **vector space** is a tuple $(V, +, \cdot)$, where V is a set together with two functions

$$+ : V \times V \longrightarrow V \qquad\qquad \cdot : \mathbb{R} \times V \longrightarrow V$$
$$(u, v) \longmapsto u + v \qquad\qquad (\lambda, v) \longmapsto \lambda v$$

such that the following properties are satisfied:

- Properties of the addition:

(A.1) $\forall u, v, w \in V$: $(u + v) + w = u + (v + w)$. (Associativity)
(A.2) $\forall u, v \in V$: $u + v = v + u$. (Commutativity)
(A.3) $\exists n \in V$, $\forall u \in V$: $n + u = u$. (Identity/neutral element of addition)
(A.4) $\forall u \in V$, $\exists v \in V$: $u + v = n$. (Inverse elements of addition)

- Compatibility of addition and scalar multiplication:

(C.1) $\forall u, v \in V$, $\lambda \in \mathbb{R}$: $\lambda \cdot (u + v) = \lambda u + \lambda v$. (Distributivity I)
(C.2) $\forall u \in V$, $\lambda, \mu \in \mathbb{R}$: $(\lambda + \mu) \cdot u = \lambda u + \mu u$. (Distributivity II)
(C.3) $\forall u \in V$, $\lambda, \mu \in \mathbb{R}$: $\lambda \cdot (\mu u) = (\lambda\mu) \cdot u$.
(C.4) $\forall u \in V$: $1 \cdot u = u$.

Exercise 1. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define for $u, v \in V$ and $\lambda \in \mathbb{R}$:

$$u \oplus v = uv,$$
$$\lambda \odot v = v^\lambda .$$

Show that $(V, \oplus, \odot)$ is a vector space.

- A polynomial functions is a function $f : \mathbb{R} \to \mathbb{R}$, such that there exist fixed $a_0, a_1, \dots, a_m \in \mathbb{R}$ with $f(x) = \sum_{j=0}^{m} a_j x^j$ for all $x \in \mathbb{R}$. The largest j with $a_j \neq 0$ is called the degree of f, denoted by $\deg(f)$.
- We denote the vector space of all polynomial functions by
$$\mathcal{P} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a polynomial function}\},$$
where the addition and scalar multiplication is the usual one given on functions $\mathbb{R} \to \mathbb{R}$.
- For $n \geq 0$ denote by $\mathcal{P}_n = \{f \in \mathcal{P} \mid \deg(f) \leq n\}$ the space of polynomial functions of degree $\leq n$.

For example, the function $f(x) = x^3 + 2x$ is an element in $\mathcal{P}_m$ for all $m \geq 3$, but not in $\mathcal{P}_2$,$\mathcal{P}_1$ or $\mathcal{P}_0$.

Exercise 2. Consider the following subset of $\mathcal{P}_2$

$$U = \{f \in \mathcal{P}_2 \mid f(1) = 0\}.$$

Find a basis of U.