Linear Algebra II

Tutorial 15

Spring 2024

25th July

$T: \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ diff. op. of order n

If $T(f) = a_0 f + a_1 f' + \dots + a_n f^{(n)} \quad (a_n \neq 0)$.

Char pol: $p_T(x) = a_0 + a_1 x + \dots + a_n x^n$.

For given $g \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$ and T we want to solve $T(f) = g$.

① Find one particular solution $T(f_p) = g$

② Determine $\ker(T)$

all solutions: $f = f_p + f_h$, $f_h \in \ker(T)$

For ②:

1) If $p_T(\lambda) = 0$ then $e^{\lambda t} \in \ker(T)$

2) $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ for pairwise different $\lambda_1, \dots, \lambda_n$ are lin. indep.

$\Rightarrow$ If $p_T(x) = c \cdot (x-\lambda_1) \cdot \ldots \cdot (x-\lambda_n) \quad c \in \mathbb{R}$ with pairwise different $\lambda_1, \dots, \lambda_n$ then $(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$ is a basis of $\ker(T)$

<u>Example:</u> Find all solutions to

$$f''(t) - 4f'(t) + 3f(t) = 2t - 1$$

with $f'(0) = f(0) = 0$.

Write $T(f) = f'' - 4f' + 3f$

$$g(x) = 2x - 1.$$

We want to solve $T(f) = g$.

① Find particular solution:

Try $f(t) = at^2 + bt + c$ for $a, b, c \in \mathbb{R}$

$$f'(t) = 2at + b$$

$$f''(t) = 2a$$

$$\Rightarrow \quad T(f) = 2a - 4(2at + b) + 3(at^2 + bt + c)$$

$$= 3a\,t^2 + (-8a+3b)\,t + 2a-4b+3c$$

$$\overset{!}{=} 2t-1$$

$$\Leftrightarrow \begin{cases} 3a = 0 & \Rightarrow a=0 \\ -8a+3b=2 & \Rightarrow b=\frac{2}{3} \\ 2a-4b+3c=-1 & \Rightarrow c=\frac{1}{3}\left(-1+\frac{8}{3}\right) \end{cases}$$

$$= \frac{5}{9}$$

$f(t) = \frac{2}{3}t + \frac{5}{9}$ is a solution.

② The char. pol. of T is

$$P_T(x) = x^2 - 4x + 3 = (x-1)(x-3).$$

$$\Rightarrow \ker(T) = \operatorname{span}\{e^t, e^{3t}\}.$$

All solutions are given by

$$f(t) = \frac{2}{3}t + \frac{5}{9} + c_1 e^{t} + c_2 e^{3t}.$$

We have $f(0) = \frac{5}{9} + c_1 + c_2$

and $f'(0) = \frac{2}{3} + c_1 + 3c_2$

If we want $f(0) = f'(0) = 0$ then

$$\begin{cases} c_1 + c_2 = -\frac{5}{9} \\ c_1 + 3c_2 = -\frac{2}{3} \end{cases} \iff \begin{matrix} c_1 = -\frac{1}{2} \\ c_2 = -\frac{1}{18} \end{matrix}$$

Our solution is

$$f(t) = \frac{2}{3}t + \frac{5}{9} - \frac{1}{2}e^{t} - \frac{1}{18}e^{3t}$$