Homework 6: Dynamical systems & Linear differential equations

---

Deadline: 29th July (23:55 JST), 2024

Exercise 1. (3+3=6 Points) Let $A \in \mathbb{R}^{2\times 2}$ be a matrix with eigenvalue $\lambda \in \mathbb{R}$, such that $\operatorname{algmu}_A(\lambda) = 2$ and $\operatorname{geomu}_A(\lambda) = 1$, i.e. A is not diagonalizable. Let $v \in \mathbb{R}^2$ be an eigenvector for the eigenvalue λ.

(i) Show that there exists a vector $w \in \mathbb{R}^2$, such that $(A - \lambda I_2)w = v$.

(ii) Show that $x(t) = te^{\lambda t}v + e^{\lambda t}w$ is a solution to the dynamical system $x'(t) = Ax(t)$, where w is the vector in (i).

Exercise 2. (6 Points) Solve the dynamical system $x'(t) = Ax(t)$, where

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Exercise 3. (4 Points) Let $F : V \to W$ be a linear map between two vector spaces V and W. Assume that $F(v) = w$ for a fixed $v \in V$ and $w \in W$. Show that the following two statements are equivalent:

(i) $F(x) = w$.

(ii) $x = v + u$ for some $u \in \ker(F)$.

Exercise 4. (6 Points) Find all solutions to the following differential equation

$$f^{(3)} - 3f'' - 10f' = 10\,,$$

such that $f(0) = f'(0) = 0$.
(Hint: First, try to find all solutions to the differential equation. For this, find one particular solution (try polynomials) and then consider the homogeneous equation. Then find those solutions out of these such that $f(0) = f'(0) = 0$ holds.)