Homework 5: Eigenvalues & Eigenvectors II

Deadline: 15th July (23:55 JST), 2024

Exercise 1. (3+2 = 5 Points)

(i) Let $U \subset \mathbb{R}^n$ be a subspace and let $P_U : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection to U. Show that P_U is diagonalizable. What are the eigenvalues of P_U?
(See Linear Algebra I Definition 13.1 for the definition of P_U)

(ii) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with a symmetric matrix $[F] = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$. Show that F has at least one eigenvalue.
(Without using Theorem 18.17, Corollary 18.18 or Lemma 18.19)

Exercise 2. (2+2+2 = 6 Points) Let $F : V \to V$ be a linear map with eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and $\dim V = n$. Show the following statements by just using Theorem 18.6 (without using Corollary 18.7 or 18.10)

(i) If $B_1, \dots, B_m$ are bases of $E_{\lambda_1}(F), \dots, E_{\lambda_m}(F)$, then $B_1 \cup \dots \cup B_m$ are linearly independent.
(Here we mean by $B_1 \cup \dots \cup B_m$ the collection of all vectors in the bases $B_1, \dots, B_m$.)

(ii) The map F is diagonalizable if and only if

$$\sum_{j=1}^{m} \dim E_{\lambda_j}(F) = n\,.$$

(iii) If F is diagonalizable then $\mathrm{geomu}_F(\lambda) = \mathrm{algmu}_F(\lambda)$ for all eigenvalues λ of F.

Exercise 3. (2+3 = 5 Points) Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}.$$

(i) Calculate all eigenvalues of A.

(ii) Find an orthogonal matrix $S \in \mathbb{R}^{3\times 3}$ such that $S^T A S$ is a diagonal matrix.

Exercise 4. (2+2+2 = 6 Points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with $A = [F]_B$, where $B = (e_1, \dots, e_n)$. Show the following three implications (without using Theorem 18.14 or Corollary 18.15)

(i) If $(F(e_1), \dots, F(e_n))$ is an orthonormal basis then A is invertible and $A^{-1} = A^T$.

(ii) If $A^{-1} = A^T$ then $F(x) \bullet F(y) = x \bullet y$ for all $x, y \in \mathbb{R}^n$.

(iii) If B_1,B_2 are orthonormal bases, then the change-of-basis matrix $S_{B_1}^{B_2}$ is orthogonal.