Homework 4: Determinant & Eigenvalues/vectors

Deadline: 24th July (23:55 JST), 2024

Exercise 1. (3+3 = 6 Points) Calculate the determinants of the following linear maps.

(i) The determinant of $F : \mathcal{P}_2 \to \mathcal{P}_2$ defined for a $p \in \mathcal{P}_2$ by

$$(F(p))(x) = p(1) + p(2)x + p(3)x^2.$$

(ii) For any $n \geq 1$ the determinant of $G : \mathcal{P}_n \to \mathcal{P}_n$ defined by $G(p) = q$, where $q(x) = xp'(x)+2p(x)$.

Exercise 2. (4+4 = 8 Points)

(i) Consider the linear map $F : \mathbb{R}^4 \to \mathbb{R}^4$, given by $F(x) = Ax$, where

$$A = \begin{pmatrix} 1 & 1 & 6 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{pmatrix}.$$

Determine all eigenvalues of F and determine for each eigenvalue λ a basis of $E_\lambda(F)$.

(ii) We define the linear map $G : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ for $p \in \mathcal{P}_2$ by

$$(G(p))(x) = (x^2 + 1)p''(x) + (x - 1)p'(x) + p(x)\,.$$

Determine all eigenvalues of G and determine for each eigenvalue λ a basis of $E_\lambda(G)$.

Exercise 3. (2+2+2+2 = 8 Points) Give examples of linear maps $F_i : \mathbb{R}^3 \to \mathbb{R}^3$ for $i = 1, 2, 3, 4$, such that

(i) F_1 has exactly three different eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

(ii) F_2 has exactly two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\dim E_{\lambda_1}(F_2) = \dim E_{\lambda_2}(F_2) = 1$.

(iii) F_3 has exactly two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\dim E_{\lambda_1}(F_3) = 1$ and $\dim E_{\lambda_2}(F_3) = 2$.

(iv) F_4 has exactly one eigenvalue $\lambda_1 \in \mathbb{R}$.

In each case calculate a basis for the eigenspaces.