Homework 3: Induction & Determinants

Deadline: 27th May (23:55 JST), 2024

Exercise 1. (2+2+2 = 6 Points) Use mathematical induction to prove the following statements.

(i) For all $n \geq 1$ we have

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 \,.$$

(ii) For all $n \geq 4$ we have $n! > 2^n$. (Here $n! = 1 \cdot 2 \cdots n$ denotes the factorial.)

(iii) Let V be a vector space which is not finitely generated. Then for any $n \geq 1$ there exist vectors $v_1, \ldots, v_n \in V$ which are linearly independent.

Exercise 2. (3+2 = 5 Points)

(i) Show (without using Proposition 17.7) that the determinant is linear in each row, i.e. for any $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ and $1 \leq l \leq n$ show that the map

$$\begin{aligned} F_{A,l} : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \det(A(l;x)) \end{aligned}$$

is linear. Here $A(l;x)$ denotes the matrix A, where the l-th row is replaced by the vector x^T. (See page 126 in the lecture notes)

(ii) Assume that A is invertible. What is the kernel of $F_{A,1}$?

Exercise 3. (4 Points) For $a_1, a_2, \ldots, a_n \in \mathbb{R}$ we define the matrix

$$A = \begin{pmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \ldots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \ldots & a_n^{n-1} \end{pmatrix} \in \mathbb{R}^{n \times n} \,.$$

Show that the determinant of A is given by

$$\det(A) = \prod_{1 \leq i < j \leq n} (a_j - a_i) \,.$$

(Hint: Use that adding a multiple of rows/columns to other rows/columns does not change the determinant (Proposition 17.6 + 17.10). Try to prove the statement then by induction on n, i.e. try to use row/column operation to find a $n-1 \times n-1$-version of such a matrix.)

Exercise 4. (1+2+2 = 5 Points) We define the matrix

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 0 & 5 & 0 \\ -1 & -2 & 3 \end{pmatrix}$$

and the polynomial $P(\lambda) = \det(A - \lambda I_3)$, where I_3 denotes the 3×3-identity matrix.

(i) Calculate $\det(A)$.

(ii) Find all solutions $\lambda \in \mathbb{R}$ to $P(\lambda) = 0$.

(iii) For each solution λ in (ii) find a non-zero vector $v \in \ker(A - \lambda I_3)$ and evaluate Av.
Can you observe a relationship between v, λ and A?