

Homework 1: Vector spaces

Deadline: 22nd April (23:55 JST), 2024

Exercise 0. (2 Points)

- (i) Try to solve the exercises below and write the solutions down by hand (paper, tablet) or by computer (Latex only). Create **one pdf-file** which contains your name on the first page and submit it before the deadline ends in TACT at the Assignment "Homework 1". Use precisely the following format as a filename: "**Familyname_Givename_LA2_HW1.pdf**". Repeat this for future Homework by replacing HW1 with HW2, HW3, etc.. Points will be removed in future homeworks if this is not the case.
- (ii) Read Chapter 14 of the lecture notes and compare the results and definitions with the corresponding results in Linear Algebra I (Chapters 1-13).

(You don't need to write down anything for Exercise 0)

Exercise 1. (3+2+2+1 = 8 Points) Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an injective function. Define on $V := \text{im}(\varphi)$ the addition \oplus and the scalar multiplication \odot for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$\begin{aligned}u \oplus v &= \varphi(\varphi^{-1}(u) + \varphi^{-1}(v)), \\ \lambda \odot v &= \varphi(\lambda \cdot \varphi^{-1}(v)).\end{aligned}$$

Here $+$ and \cdot denote the usual addition and multiplication in \mathbb{R} .

- (i) Show that (V, \oplus, \odot) is a vector space. What is the neutral element of (V, \oplus, \odot) ? (i.e. check that the operations \oplus and \odot satisfy the properties (A.1) – (A.4) and (C.1) – (C.4).)
- (ii) Determine all subspaces of (V, \oplus, \odot) .
- (iii) Find an isomorphism

$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot).$$

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space \mathbb{R}^1 with the usual addition and multiplication of real numbers.

- (iv) Do (ii) and (iii) explicitly for the case $\varphi(x) = e^x$.

Exercise 2. (2+2+2+2 = 8 Points) Let \mathcal{P} denote the set of all polynomial functions from \mathbb{R} to \mathbb{R} . Define the following subsets

$$\begin{aligned}\mathcal{P}_3 &= \{f \in \mathcal{P} \mid \deg(f) \leq 3\}, \\ U &= \{f \in \mathcal{P}_3 \mid f(-2) = f(0) = 0\} \subset \mathcal{P}_3.\end{aligned}$$

- (i) Show that U is a subspace of \mathcal{P}_3 .
- (ii) Determine a basis $B = (b_1, \dots, b_n)$ of U .
- (iii) Determine the coordinate vector $[f]_B$ for the function $f \in U$ given by $f(x) = x(x+2)^2$.
- (iv) Extend the basis B to a basis \tilde{B} of \mathcal{P}_3 . (i.e. find a basis of \mathcal{P}_3 , which contains all the basis elements of your basis B of U)

Exercise 3. (2+2+2 = 6 Points) Define for $M \in \mathbb{R}^{2 \times 2}$ the following set

$$C(M) = \{A \in \mathbb{R}^{2 \times 2} \mid AM = MA\}.$$

- (i) Show that for a given fixed $M \in \mathbb{R}^{2 \times 2}$ the set $C(M)$ is a subspace of $\mathbb{R}^{2 \times 2}$.
- (ii) For $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.
- (iii) Show that for all $M \in \mathbb{R}^{2 \times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4.$$

(i.e. show that there exists no matrix M , such that $C(M)$ has dimension 0 or 1.)

