Linear Algebra II & Mathematics Tutorial 2b Spring 2024
Nagoya University, G30 Program Instructor: Henrik Bachmann

Final exam

Good luck Viel Erfolg 好好儿考啊 시험 잘봐 Semoga sukses амжилт хүсье
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Exercise 1. (14 Points) We define the following matrix

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}.$$

(i) Compute the determinant of A.

(ii) Find all eigenvalues of A. For each eigenvalue λ determine a basis of the eigenspace $E_\lambda(A)$.

(iii) Is A diagonalizable and/or invertible and/or orthogonal? Justify your answers.

(iv) Find an invertible matrix S such that $S^{-1}AS$ is a diagonal matrix where the top left entry and the bottom right entry are the same.

Exercise 2. (11 Points) Decide if the following statements are true or false. Justify your answer by giving a short explanation.

(i) The set $U = \left\{A \in \mathbb{R}^{2\times 2} \mid A^T A = 0\right\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

(ii) The set $U = \{p \in \mathcal{P}_2 \mid p(x) = p(2x) \text{ for all } x \in \mathbb{R}\}$ is a subspace of $\mathcal{P}_2$.

(iii) If $A, B \in \mathbb{R}^{n\times n}$ are symmetric and have the same eigenvalues and eigenspaces, then $A = B$.

(iv) For any $A \in \mathbb{R}^{n\times n}$ the matrix $A^T - A$ is diagonalizable.

(v) If there exist injective linear maps $F : V \to W$ and $G : W \to V$ then $V \cong W$.
(Here we assume that V and W are finitely generated vector spaces)

Exercise 3. (9 Points) Give an example of

(i) a $A \in \mathbb{R}^{2\times 2}$ such that $x(t) = \begin{pmatrix} e^t + e^{2t} \\ e^{2t} \end{pmatrix}$ is a solution to the dynamical system $x'(t) = Ax(t)$.

(ii) a differential operator T of order 4 with $\ker(T) = \operatorname{span}\{e^t\cos(3t), e^t, e^t\sin(3t), e^{2t}\}$.

(iii) a linear map $H : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ with $\operatorname{im}(H) = \ker(H)$.

Justify your examples.

Exercise 4. (8 Points) Let $x_n \in \mathbb{R}^2$ for $n \geq 0$ be defined by

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad x_{n+1} = Mx_n\,, \qquad \text{where} \qquad M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Determine an explicit formula for x_n.

Exercise 5. (8 Points) Find all functions $f \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$2f(t) = f'''(t) + f''(t) + t$$

and $f(0) = f'(0) = 0$.