Linear Algebra II - Midterm

Spring 2023

Nagoya University, G30 Program

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Total: 36 Points

1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.

(i) The matrix $A_x = \begin{pmatrix} x & -2x \\ 3 & 3x \end{pmatrix}$ is invertible for any $x \in \mathbb{R}$ with $x \neq 0$.

(ii) If V, W are two vector spaces with $\dim(V) = 3$ and $\dim(W) = 2$ then there exists an injective linear map $F : V \to W$.

(iii) If $F : \mathbb{R}^n \to \mathbb{R}^n$ is a surjective linear map then $\det(F) \neq 0$.

(iv) If (f_1, f_2, f_3) is a basis of $\mathcal{P}_2$ then $(f_1 + f_2', f_2 + f_3', f_3 + f_1')$ is a basis of $\mathcal{P}_2$.

2) (5+3=8 Points) Consider the bases $B = (2x + 1, x + 1)$ of $\mathcal{P}_1$ and $D = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix})$ of $\mathbb{R}^2$, the linear map $F : \mathcal{P}_1 \to \mathbb{R}^2$ with

$$[F]_B^D = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $g \in \mathcal{P}_1$ defined by $g(x) = x - 1$.
(You do <u>not</u> need to show that B and D are bases)

(i) Determine $F(g)$.

(ii) Let $c_B : \mathbb{R}^2 \to \mathcal{P}_1$ be the coordinate map with respect to the basis B. Consider the linear map $G = F \circ c_B : \mathbb{R}^2 \to \mathbb{R}^2$ and determine its matrix $[G]$.

3) (3+4=7 Points) For $\mu \in \mathbb{R}$ let $U_\mu = \{f \in \mathcal{P}_3 \mid f(\mu x) = \mu f(x) \text{ for all } x \in \mathbb{R}\}$.

(i) Show that U_μ is a subspace of $\mathcal{P}_3$ for any $\mu \in \mathbb{R}$.

(ii) Determine a basis of U_μ for any $\mu \in \mathbb{R}$.

4) (3+3+2=8 Points) We define the following elements in $\mathbb{R}^{2\times 2}$

$$m_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\ m_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ m_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ m_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and define the following linear map

$$T : \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$$
$$A \longmapsto A + A^T.$$

(You do <u>not</u> need to show that T is linear)

(i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2\times 2}$.

(ii) Determine $[T]_M$ and $\det(T)$.

(iii) Calculate $\dim(\ker(T))$ and $\dim(\mathrm{im}(T))$.

5) (2+3=5 Points) We define for $n \geq 1$ the following number

$$C(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)}.$$

(i) Evaluate $C(1), C(2)$ and $C(3)$. Make a guess for an explicit formula for $C(n)$.

(ii) Prove your guess by using induction.