

Linear Algebra II

Lecture 11: Calculating Eigenvalues & Python & Population behaviour

Henrik Bachmann

Nagoya University, 3rd July June 2023

Download these slides at https://www.henrikbachmann.com/la2_2023.html.

There you can also find a link to a Google Colab notebook, which you can use during the lecture

Recall: Symmetric linear maps

Definition 5.16

- An **eigenbasis** of a linear map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a basis consisting of eigenvectors of F .
- Let $U \subset \mathbb{R}^n$ be a subspace. A linear map $F : U \rightarrow U$ is called **symmetric** if we have for all $x, y \in U$

$$x \bullet F(y) = F(x) \bullet y.$$

We saw last lecture:

$$F \text{ is symmetric} \iff [F]_B \text{ is a symmetric matrix for an orthonormal basis } B.$$

Recall: Spectral theorem

Theorem 5.17 (Spectral Theorem)

Let $U \subset \mathbb{R}^n$ be a subspace and $F : U \rightarrow U$ a linear map. Then F is symmetric if and only if there exists an orthonormal eigenbasis of F .

As a special case we have the following:

Corollary 5.18

A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if and only if there exists an orthogonal matrix $S \in \mathbb{R}^{n \times n}$, such that

$$S^T A S = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix},$$

for some $d_1, \dots, d_n \in \mathbb{R}$.

Recall: The reflection example

In the lecture we considered the matrix

$$A = \frac{1}{3} \begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$$

and calculated its eigenvalues and eigenvectors. We saw that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. The eigenspaces are

$$E_1(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad E_{-1}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Question

How can we find an orthogonal matrix S , such that

$$S^T A S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}?$$

Recall: The reflection example

$$E_1(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad E_{-1}(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Answer: To get an orthonormal eigenbasis one can apply Gram-Schmidt to the bases of each eigenspace:

$$E_1(A) = \text{span} \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right\}, \quad E_{-1}(A) = \text{span} \left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

$$\text{Setting } S = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \quad \text{we get } S^T A S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Question

How can you calculate eigenvalues for big matrices? Calculating the characteristic polynomial of a 100×100 matrix and then finding its zeros sounds difficult. (it is!)

Question

How can you calculate eigenvalues for big matrices? Calculating the characteristic polynomial of a 100×100 matrix and then finding its zeros sounds difficult. (it is!)

Rough idea: If we have a matrix A and a random vector v , then $A^k v$ for large k will give an approximation for an eigenvector of A for its "largest" eigenvalue.

Question

How can you calculate eigenvalues for big matrices? Calculating the characteristic polynomial of a 100×100 matrix and then finding its zeros sounds difficult. (it is!)

Rough idea: If we have a matrix A and a random vector v , then $A^k v$ for large k will give an approximation for an eigenvector of A for its "largest" eigenvalue.

More precisely we consider the following situation:

- Let $A \in \mathbb{R}^{n \times n}$ be diagonalizable
- Eigenvalues of A : $\lambda_1, \dots, \lambda_n$ with $\lambda_1 = \dots = \lambda_d$ and $|\lambda_1| > |\lambda_j|$ for $j = d + 1, \dots, n$.
(Here λ_1 is called the **dominant eigenvalue** of A).
- Let (b_1, \dots, b_n) be a basis of \mathbb{R}^n of eigenvectors b_j with eigenvalue λ_j .
- Suppose we have $v = \sum_{j=1}^n \alpha_j b_j$ for $\alpha_j \in \mathbb{R}$, such that $\sum_{j=1}^d \alpha_j b_j \neq 0$.

Power iteration

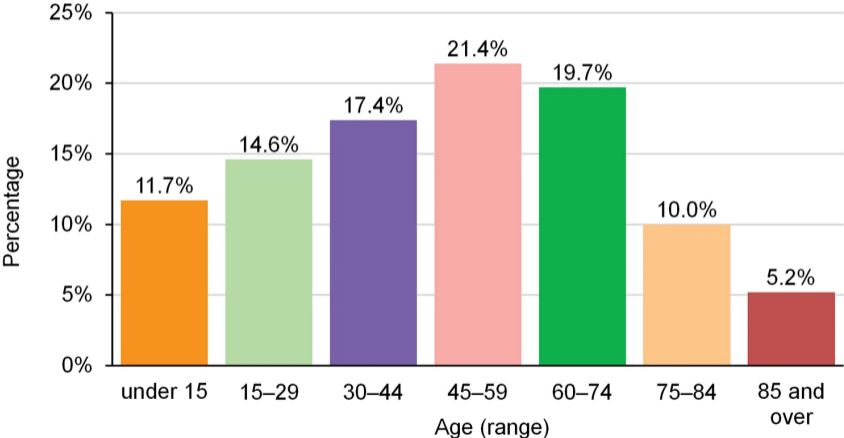
Let $A \in \mathbb{R}^{n \times n}$

- Start with a random vector $v_0 \in \mathbb{R}^n$.
- Define for $k \geq 0$

$$v_{k+1} = \frac{Av_k}{\|Av_k\|}$$

Then for large k the vector v_k will give an approximation for an eigenvector of A for its dominant eigenvalue.

Japan age breakdown (2022*)



© Encyclopædia Britannica, Inc.

*June.

Consider the following Leslie matrix

$$C = \begin{pmatrix} 0 & 3 & 2 \\ \frac{4}{5} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \end{pmatrix}$$

which models the population behavior of a population with 3 age groups.

Interpretation:

- Three age groups (a_1, a_2, a_3)
- a_1 does not breed and has a survival rate of 80% $(\frac{4}{5})$
- a_2 makes 3 babies (per cycle) and has a survival rate of 40% $(\frac{2}{5})$
- a_3 makes 2 babies (per cycle) and dies out after one generation.