

Homework 2: Linear maps

Deadline: 14th May (23:55 JST), 2023

Exercise 1. (1+1+2+2 = 6 Points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}. \quad (n \geq 2)$$

In this exercise we want to prove the following explicit formula

$$F_n = \frac{1}{2^n \sqrt{5}} \left((1 + \sqrt{5})^n - (1 - \sqrt{5})^n \right). \quad (0.1)$$

For this follow the following steps:

- (i) Find a linear map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, such that $F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for $n \geq 1$, where $F^n = \underbrace{F \circ \dots \circ F}_n$.
- (ii) We define the following two bases of \mathbb{R}^2 :

$$B_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad B_2 = \left(\begin{pmatrix} 2 \\ 1 + \sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1 - \sqrt{5} \end{pmatrix} \right).$$

Determine the change-of-basis matrices $S_{B_1}^{B_2}$ and $(S_{B_1}^{B_2})^{-1}$.

- (iii) Calculate $[F]_{B_1}$ and $[F]_{B_2}$.
- (iv) Calculate $[F]_{B_1}^n$ by using

$$[F]_{B_1} = (S_{B_1}^{B_2})^{-1} [F]_{B_2} S_{B_1}^{B_2}$$

and prove (0.1) by using i).

Exercise 2. (2+2+2 = 6 Points) We define the space of Fibonacci sequences by

$$\mathcal{F} = \{(a_n)_{n \geq 0} \in \mathcal{J} \mid a_n = a_{n-1} + a_{n-2} \text{ for all } n \geq 2\},$$

where \mathcal{J} denotes the vector space of all infinite sequences (see Lecture 1).

- (i) Show that \mathcal{F} is a vector space with the addition and scalar multiplication coming from \mathcal{J} .
- (ii) Show that \mathcal{F} is finitely generated and find a basis B of \mathcal{F} .
- (iii) Define the map $G : \mathcal{F} \rightarrow \mathcal{F}$ on a sequence $a = (a_n)_{n \geq 0}$ by $G(a) = b$, where $b = (b_n)_{n \geq 0}$ is given by $b_n = a_{n+2}$. Show that G is an isomorphism and determine $[G]_B$, where B is the basis in (ii).

Exercise 3. (2+2+2+2 = 8 Points) For $n \geq 0$ we define the map $H_n : \mathcal{P}_n \rightarrow \mathbb{R}^3$ for a $p \in \mathcal{P}_n$ by

$$H_n(p) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}.$$

- (i) Show that H_n is a linear map.
- (ii) Show that H_2 is an isomorphism and calculate the inverse of H_2 .
- (iii) Check if H_0, H_1 and H_3 are injective and/or surjective.
- (iv) Determine a basis of $\ker(H_3)$.