Homework 1: Vector spaces

Deadline: 23rd April (23:55 JST), 2023

Exercise 0. (2 Points)

(i) Try to solve the exercises below and write the solutions down by hand (paper, tablet) or by computer (Latex only). Create **one pdf-file** which contains your name on the first page and submit it before the deadline ends in TACT at the Assignment "Homework 1". Use precisely the following format as a filename: "**Familyname_Givenname_LA2_HW1.pdf**". Repeat this for future Homework by replacing HW1 with HW2, HW3, etc. (Points will be removed in future homeworks if this is not the case)

(ii) Read Section 1 of the overview notes and compare the results and definitions with the corresponding results in Linear Algebra I.

(You don't need to write down anything for Exercise 0)

Exercise 1. (4+2+2 = 8 Points) Let $V = \mathbb{R}$ be the set of real numbers. Define on V the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$u \oplus v = u + v - 2023\,,$$
$$\lambda \odot v = \lambda(v - 2023) + 2023\,.$$

(i) Show that $(V, \oplus, \odot)$ is a vector space. What is the neutral element in V?
(i.e. check that the operations $\oplus$ and $\odot$ satisfy the properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$.)

(ii) Determine all subspaces of $(V, \oplus, \odot)$.

(iii) Find an isomorphism

$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot)\,.$$

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

Exercise 2. (2+2+2 = 6 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \left\{A \in \mathbb{R}^{2\times 2} \mid AM = MA\right\}\,.$$

(i) Show that for a given fixed $M \in \mathbb{R}^{2\times 2}$ the set $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$.

(ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

(iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4\,.$$

(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)

Exercise 3. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\mathcal{P}_3 = \{f \in \mathcal{P} \mid \deg(f) \leq 3\}\,,$$
$$U = \{f \in \mathcal{P}_3 \mid f(-1) = f(1) = 0\} \subset \mathcal{P}_3\,.$$

(i) Show that U is a subspace of $\mathcal{P}_3$.

(ii) Determine a basis $B = (b_1, \ldots, b_n)$ of U.

(iii) Determine the coordinate vector $[f]_B$ for the function $f \in U$ given by $f(x) = (x + 1)x(x - 1)$.

(iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)