

Linear Algebra II

Overview notes

G30 Program, Nagoya University (Spring 2022)

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Lecture notes and exercises are available at: https://www.henrikbachmann.com/la2_2022.html

These notes serve as a compact overview of the definitions, propositions, lemmas, corollaries, and theorems given in the lectures. This is Version 2 from June 9, 2022. The proofs, examples, and explanations are provided in the handwritten notes/lectures. The reference book for this course is [B], and we will probably cover Chapters 4,6,7 and 9 during this semester.

If you find any typos in this note, please let me know!

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References

[B] O. Bretscher: *Linear Algebra with Applications*, 4th edition, Pearson 2009.

1 Vector spaces

Definition 1.1. A (real) vector space (linear space) is a set V together with two functions

<p style="margin: 0;"><i>Addition</i></p> $+ : V \times V \longrightarrow V$ $(u, v) \longmapsto u + v$	<p style="margin: 0;"><i>Scalar multiplication</i></p> $\cdot : \mathbb{R} \times V \longrightarrow V$ $(\lambda, v) \longmapsto \lambda \cdot v = \lambda v$
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satisfying the following properties:

- *Properties of the addition:*

- (A.1) $\forall u, v, w \in V: (u + v) + w = u + (v + w)$. (*Associativity*)
- (A.2) $\forall u, v \in V: u + v = v + u$. (*Commutativity*)
- (A.3) $\exists n \in V, \forall u \in V: n + u = u$. (*Identity/neutral element of addition*)
- (A.4) $\forall u \in V, \exists v \in V: u + v = n$. (*Inverse elements of addition*)

- *Compatibility of addition and scalar multiplication:*

- (C.1) $\forall u, v \in V, \lambda \in \mathbb{R}: \lambda \cdot (u + v) = \lambda u + \lambda v$. (*Distributivity I*)
- (C.2) $\forall u \in V, \lambda, \mu \in \mathbb{R}: (\lambda + \mu) \cdot u = \lambda u + \mu u$. (*Distributivity II*)
- (C.3) $\forall u \in V, \lambda, \mu \in \mathbb{R}: \lambda \cdot (\mu u) = (\lambda \mu) \cdot u$.
- (C.4) $\forall u \in V: 1 \cdot u = u$.

We write $(V, +, \cdot)$ for the vector space V if we want to emphasize which addition and scalar multiplication we are using.

Proposition 1.2. Let V be a vector space and $u \in V$.

- i) $u + n = u$.
- ii) If $n, \tilde{n} \in V$ both satisfy (A.3) in Definition 1.1, then $n = \tilde{n}$.
(The Identity element is unique)
- iii) If for a fixed $u \in V$ the elements $v, \tilde{v} \in V$ both satisfy (A.4), i.e. $u + v = u + \tilde{v} = n$, then $v = \tilde{v}$.
(The inverse of an element u is unique)
- iv) $u + (-1)u = n$.

The identity (also called neutral) element $n \in V$ of a vector space is usually (by abuse of notation) also denoted by 0. Be always aware in the following if 0 means the real number 0 or the identity element of a vector space. (These are two different things!)

Definition 1.3. Let V be a vector space. A subset $U \subset V$ is a **subspace** if

- i) $0 \in U$.
- ii) $\forall u, v \in U: u + v \in U$.
- iii) $\forall u \in U, \lambda \in \mathbb{R}: \lambda u \in U$.

Proposition 1.4. If $U \subset V$ is a subspace, then U is also a vector space with the operations inherited from V .

Definition 1.5. Let V be a vector space and $v_1, \dots, v_n \in V$.

- i) The **span** of the elements v_1, \dots, v_n is given by the set of all their **linear combinations**, i.e.

$$\text{span}\{v_1, \dots, v_n\} = \left\{ \sum_{i=1}^n \lambda_i v_i \in V \mid \lambda_1, \dots, \lambda_n \in \mathbb{R} \right\}.$$

- ii) The elements v_1, \dots, v_n **span (or generate) the space** V if $\text{span}\{v_1, \dots, v_n\} = V$.
- iii) V is **finitely generated** if there exist $v_1, \dots, v_n \in V$ with $\text{span}\{v_1, \dots, v_n\} = V$.
(i.e. one just needs finitely many elements to generate the space)
- iv) The elements v_1, \dots, v_n are **linearly independent** if

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \implies \lambda_1 = \dots = \lambda_n = 0.$$

- v) $B = (v_1, \dots, v_n)$ is a **basis** of V if v_1, \dots, v_n are linearly independent and $\text{span}\{v_1, \dots, v_n\} = V$.

Proposition 1.6. Let V be a vector space and $v_1, \dots, v_n \in V$. The following statements are equivalent.

- i) v_1, \dots, v_n are linearly dependent.
- ii) There exist a $1 \leq j \leq n$ such that $v_j \in \text{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$.
- iii) There exist a $1 \leq j \leq n$ such that $\text{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\} = \text{span}\{v_1, \dots, v_n\}$.

Lemma 1.7. If $v_1, \dots, v_l \in V$ are linearly independent and $V = \text{span}\{w_1, \dots, w_m\}$, then $l \leq m$.

Theorem 1.8. Let V be a finitely generated vector space. Then we have the following

- i) V has a (finite) basis.
- ii) All bases of V have the same number of elements.
- iii) If $v_1, \dots, v_l \in V$ are linearly independent then there exist $v_{l+1}, \dots, v_n \in V$ such that (v_1, \dots, v_n) is a basis of V .
- iv) If $V = \text{span}\{w_1, \dots, w_m\}$, then there exist a subset $\{u_1, \dots, u_l\} \subset \{w_1, \dots, w_m\}$, such that (u_1, \dots, u_l) is a basis of V .

Definition 1.9. Let V be a finitely generated vector space with basis (v_1, \dots, v_n) . Then $\dim(V) = n$ is the **dimension of V** .

Corollary 1.10. Let V be a vector space with $\dim(V) = n$ and $v_1, \dots, v_n \in V$. Then the following statements are equivalent.

- i) v_1, \dots, v_n are linearly independent.
- ii) $V = \text{span}\{v_1, \dots, v_n\}$.
- iii) (v_1, \dots, v_n) is a basis of V .

Proposition 1.11. Let V be finitely generated and $U \subset V$ a subspace. Then U is also finitely generated.

Proposition 1.12. Let $B = (v_1, \dots, v_n)$ be a basis of V . Then for all $u \in V$ there exist unique $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, such that

$$u = \sum_{i=1}^n \lambda_i v_i.$$

Definition 1.13. Let $B = (v_1, \dots, v_n)$ be a basis of V .

- i) The $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ in Proposition 1.12 are called the **coordinates** of $u \in V$ in the basis B .
- ii) The vector $[u]_B \in \mathbb{R}^n$ given by

$$[u]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is called the **coordinate vector** of u with respect to the basis B .

2 Linear maps

Definition 2.1. Let V, W be vector spaces. A **linear map** is a function $F : V \rightarrow W$ satisfying

- i) $F(u + v) = F(u) + F(v)$ for all $u, v \in V$.
- ii) $F(\lambda \cdot u) = \lambda \cdot F(u)$ for all $u \in V, \lambda \in \mathbb{R}$.

Definition 2.2. Let $F : V \rightarrow W$ be a linear map.

- i) The **kernel of F** is given by

$$\ker(F) = \{u \in V \mid F(u) = 0\} \subset V.$$

- ii) The **image of F** is given by

$$\text{im}(F) = \{w \in W \mid \exists u \in V : w = F(u)\} \subset W.$$

With the same arguments as in the \mathbb{R}^n -case we see that $\ker(F)$ is a subspace of V and $\text{im}(F)$ is a subspace of W . If $\text{im}(F)$ is finitely generated, we define the **rank of F** by $\text{rk}(F) = \dim(\text{im}(F))$.

Theorem 2.3 (kernel-image theorem). *Let V be finitely generated and let $F : V \rightarrow W$ be a linear map to an arbitrary vector space W . Then*

$$\dim V = \dim(\ker(F)) + \dim(\text{im}(F)).$$

Definition 2.4. *i) (Recall) A function $f : X \rightarrow Y$ is **invertible** if there exist a function $g : Y \rightarrow X$ such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$. f is invertible iff f is bijective, i.e. injective and surjective.*

*ii) An invertible linear map $F : V \rightarrow W$ is called an **isomorphism**.*

*iii) Two vector spaces V and W are called **isomorphic** (Notation: $V \cong W$) if there exists an isomorphism $F : V \rightarrow W$.*

Theorem 2.5. *i) A linear map $F : V \rightarrow W$ is an isomorphism iff $\ker(F) = \{0\}$ (F is injective) and $\text{im}(F) = W$ (F is surjective).*

ii) Let $F : V \rightarrow W$ be an isomorphism and (b_1, \dots, b_n) a basis of V . Then $(F(b_1), \dots, F(b_n))$ is a basis of W .

iii) Let V, W be finitely generated and $V \cong W$ then $\dim(V) = \dim(W)$.

iv) Let V, W be finitely generated and $\dim(V) = \dim(W)$. Then for a linear map $F : V \rightarrow W$ the following three statements are equivalent

(a) F is an isomorphism.

(b) $\ker(F) = \{0\}$.

(c) $\text{im}(F) = W$.

Proposition 2.6. *Let V be finitely generated with basis $B = (b_1, \dots, b_n)$, i.e. $\dim(V) = n$. Then the **coordinate map***

$$c_B : \mathbb{R}^n \longrightarrow V, \\ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \longmapsto \sum_{i=1}^n \lambda_i b_i$$

is an isomorphism. The inverse is given by $c_B^{-1}(u) = [u]_B$ for $u \in V$.

Corollary 2.7. *Let V, W be finitely generated. Then the following two statements are equivalent*

i) $V \cong W$.

ii) $\dim(V) = \dim(W)$.

3 The matrix of a linear map

In the following V and W are finitely generated vector spaces.

Definition 3.1. Let $B_V = (v_1, \dots, v_n)$ be a basis of V , $B_W = (w_1, \dots, w_m)$ be a basis of W and let $F : V \rightarrow W$ be a linear map. The **matrix of F with respect to B_V and B_W** is defined by

$$[F]_{B_V}^{B_W} = [c_{B_W}^{-1} \circ F \circ c_{B_V}].$$

Here $c_{B_W}^{-1} \circ F \circ c_{B_V}$ is the linear map from \mathbb{R}^n to \mathbb{R}^m for which the corresponding matrix was defined before. We have the following diagram

$$\begin{array}{ccc} V & \xrightarrow{F} & W \\ c_{B_V} \uparrow & & \downarrow c_{B_W}^{-1} \\ \mathbb{R}^n & \xrightarrow{c_{B_W}^{-1} \circ F \circ c_{B_V}} & \mathbb{R}^m \end{array}$$

We have

$$[F]_{B_V}^{B_W} = \left(\begin{array}{c|ccc|c} & & & & \\ \hline [F(v_1)]_{B_W} & \cdots & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} \right).$$

In other words: The j -th column of $[F]_{B_V}^{B_W}$ is given by the vector $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$, where $F(v_j) = \sum_{i=1}^m \lambda_i w_i$.

Definition 3.2. Let $B_1 = (v_1, \dots, v_n)$ and $B_2 = (u_1, \dots, u_n)$ be bases of V . The **change-of-basis matrix from B_1 to B_2** is the matrix

$$S_{B_1}^{B_2} = [\text{id}_V]_{B_1}^{B_2} = [c_{B_2}^{-1} \circ c_{B_1}] = \left(\begin{array}{c|ccc|c} & & & & \\ \hline [v_1]_{B_2} & \cdots & & & \\ \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} \right).$$

4 Determinants

Definition 4.1. A **pattern** in an $n \times n$ -matrix is a choice of entries, such that precisely one entry from each row and each column is chosen.

Definition 4.2. i) A bijective map $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ is called a **permutation** of $\{1, \dots, n\}$.

ii) S_n denotes the set of all permutations of $\{1, \dots, n\}$.

Patterns in an $n \times n$ -matrix corresponds exactly to the permutations of $\{1, \dots, n\}$. For each $\sigma \in S_n$ we have the pattern

$$P = \{(1, \sigma(1)), (2, \sigma(2)), \dots, (n, \sigma(n))\},$$

where (i, j) denotes the choice of the i -th row and the j -th column.

Definition 4.3. i) The **number of inversion** of a permutation $\sigma \in S_n$, denoted by $\text{inv}(\sigma)$, is the number of pairs $(i, \sigma(i)), (j, \sigma(j))$ with $i < j$ and $\sigma(i) > \sigma(j)$.

ii) The **sign** of a permutation $\sigma \in S_n$ is defined by

$$\text{sign}(\sigma) = (-1)^{\text{inv}(\sigma)}.$$

Definition 4.4. The **determinant** of a $n \times n$ -matrix $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ is defined by

$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{i=1}^n a_{i, \sigma(i)}.$$

4.1 Properties of determinants

Lemma 4.5. For all $\sigma \in S_n$ we have $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$.

Proposition 4.6. For any $A \in \mathbb{R}^{n \times n}$ we have $\det(A) = \det(A^T)$.

For $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ define for a vector $x \in \mathbb{R}^n$ and $1 \leq l \leq n$ the matrix $A(l; x)$ as the matrix where the l -th row of A gets replaced by x , i.e.

$$A(l; x) = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & & & \vdots \\ a_{l-1,1} & a_{l-1,2} & \cdots & a_{l-1,n} \\ x_1 & x_2 & \cdots & x_n \\ a_{l+1,1} & a_{l+1,2} & \cdots & a_{l+1,n} \\ \vdots & & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \in \mathbb{R}^{n \times n}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Proposition 4.7. For any $A = (a_{i,j}) \in \mathbb{R}^{n \times n}$ and $1 \leq l \leq n$ the map

$$F_{A,l} : \mathbb{R}^n \longrightarrow \mathbb{R} \\ x \longmapsto \det(A(l; x))$$

is a linear map, i.e. the determinant is linear in each row,

Proposition 4.8. For $A \in \mathbb{R}^{n \times n}$ let $B \in \mathbb{R}^{n \times n}$ be a matrix obtained from the matrix A by swapping two rows. Then we have

$$\det(A) = -\det(B).$$

Corollary 4.9. If a matrix $A \in \mathbb{R}^{n \times n}$ contains two equal rows or columns, then $\det(A) = 0$.

Recall from Linear Algebra I that there are three types of **row operations** for a matrix $A \in \mathbb{R}^{n \times n}$. ($1 \leq i, j \leq n, i \neq j, \lambda \in \mathbb{R}$).

(R1) Add λ -times the j -th row to the i -th row.

(R2) For $\lambda \neq 0$ multiply the i -th row with λ .

(R3) Swap the j -th row with the i -th row.

Two matrices $A, B \in \mathbb{R}^{n \times n}$ are called **row equivalent**, if one can obtain B from A by using the row operations (R1), (R2) and (R3). Notation: $A \sim B$.

Proposition 4.10. Let $A, B \in \mathbb{R}^{n \times n}$.

i) If B is obtained from A by using (R1), then $\det(B) = \det(A)$.

ii) If B is obtained from A by using (R2), then $\det(B) = \lambda \det(A)$.

iii) If B is obtained from A by using (R3), then $\det(B) = -\det(A)$.

Theorem 4.11. A matrix $A \in \mathbb{R}^{n \times n}$ is invertible if and only if $\det(A) \neq 0$.

Theorem 4.12. i) For all $A, B \in \mathbb{R}^{n \times n}$ we have $\det(AB) = \det(A) \det(B)$.

ii) If A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

Corollary 4.13. Let V be a finitely generated vector space, $F : V \rightarrow V$ a linear map and B_1, B_2 two bases of V . Then

$$\det([F]_{B_1}) = \det([F]_{B_2}),$$

where $[F]_B = [F]_B^B$ denotes the matrix of F with respect to the basis B (Definition 3.1).

Definition 4.14. Let V be a finitely generated vector space, $F : V \rightarrow V$ a linear map and B any basis of V . We define the **determinant of the linear map F** by

$$\det(F) = \det([F]_B).$$

Notice that $v = 0$ always satisfies (5.1) for any $\lambda \in \mathbb{R}$, since F is a linear map. This is one of many reasons why $v = 0$ is not called an eigenvector of F .

In the following, we always assume that V is a finitely generated vector space.

Definition 5.2. Let $F : V \rightarrow V$ be a linear map let $\text{id}_V : V \rightarrow V$ be the identity map on V .

- i) The polynomial $f_F(\lambda) = \det(F - \lambda \text{id}_V)$ is called the **characteristic polynomial** of F .
- ii) Let $\lambda \in \mathbb{R}$ be an eigenvalue of F . Then the space

$$\begin{aligned} E_\lambda(F) &= \ker(F - \lambda \text{id}_V) \\ &= \{v \in V \mid F(v) = \lambda v\} \end{aligned}$$

is called the **eigenspace** of F with respect to the eigenvalue λ .

The eigenspace $E_\lambda(F)$ contains therefore all eigenvectors of F with eigenvalue λ and the zero vector.

Definition 5.3. i) Let $\dim V = n$. A linear map $F : V \rightarrow V$ is called **diagonalizable** if there exist a basis B of V , such that

$$[F]_B = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

for some $d_1, \dots, d_n \in \mathbb{R}$.

- ii) A matrix $A \in \mathbb{R}^{n \times n}$ is called **diagonalizable** if there exists an invertible matrix $S \in \mathbb{R}^{n \times n}$ with

$$S^{-1}AS = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

for some $d_1, \dots, d_n \in \mathbb{R}$.

Lemma 5.4. Let B be a basis of V and let $F : V \rightarrow V$ be a linear map. Then the following two statements are equivalent

- i) The linear map F is diagonalizable.
- ii) The matrix $[F]_B$ is diagonalizable.

Lemma 5.5. Let $F : V \rightarrow V$ be a linear map and $B = (b_1, \dots, b_n)$ be a basis of V , such that all b_i are eigenvectors of F , i.e. $F(b_i) = d_i b_i$ for some $d_i \in \mathbb{R}$ and $i = 1, \dots, n$. Then F is diagonalizable and

$$[F]_B = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

Conversely, if F is diagonalizable then there exists a basis of eigenvectors.

Theorem 5.6. Let $v_1, \dots, v_m \in V$ be eigenvectors of a linear map $F : V \rightarrow V$ with different eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Then v_1, \dots, v_m are linearly independent.

Corollary 5.7. Let $F : V \rightarrow V$ be a linear map with eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and $\dim V = n$.

i) If F has n distinct eigenvalues, i.e. $m = n$, then F is diagonalizable.

ii) If B_1, \dots, B_m are bases of $E_{\lambda_1}(F), \dots, E_{\lambda_m}(F)$, then $B_1 \cup \dots \cup B_m$ are linearly independent.

iii) The map F is diagonalizable if and only if

$$\sum_{j=1}^m \dim E_{\lambda_j}(F) = n.$$

Definition 5.8. Let $F : V \rightarrow V$ be a linear map and let $\lambda \in \mathbb{R}$ be an eigenvalue of F .

i) The **algebraic multiplicity** of λ , denoted by $\text{algnu}_F(\lambda)$, is the multiplicity of λ in the characteristic polynomial f_F .

ii) The **geometric multiplicity** of λ is given by $\text{geomu}_F(\lambda) = \dim E_{\lambda}(F)$.

Theorem 5.9. Let $F : V \rightarrow V$ be a linear map and $\lambda \in \mathbb{R}$ be an eigenvalue of F . Then

$$\text{geomu}_F(\lambda) \leq \text{algnu}_F(\lambda).$$

Corollary 5.10. If F is diagonalizable then $\text{geomu}_F(\lambda) = \text{algnu}_F(\lambda)$ for all eigenvalues λ of F .

5.1 The spectral theorem

In this section we will just consider the vector space $V = \mathbb{R}^n$. Recall that the **norm** of a vector $x \in \mathbb{R}^n$ is defined by

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Definition 5.11. An **orthogonal map** is a linear map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, such that

$$\|F(x)\| = \|x\|, \quad \forall x \in \mathbb{R}^n,$$

i.e. the map F does not change the norm of a vector. We call a matrix $A \in \mathbb{R}^{n \times n}$ **orthogonal** if $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.

Recall that the **dot product** \bullet for two vectors $x, y \in \mathbb{R}^n$ is defined by

$$x \bullet y = x^T y = x_1 y_1 + \dots + x_n y_n \in \mathbb{R}.$$

With this the norm of a vector can also be written as $\|x\| = \sqrt{x \bullet x}$.

Lemma 5.12. For all $x, y \in \mathbb{R}^n$ we have

$$x \bullet y = \frac{1}{2} (\|x + y\|^2 - \|x\|^2 - \|y\|^2) .$$

Proposition 5.13. A linear map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal if and only if

$$F(x) \bullet F(y) = x \bullet y$$

for all $x, y \in \mathbb{R}^n$.

Recall: We say that x and y are **orthogonal** if $x \bullet y = 0$. A basis $B = (b_1, \dots, b_n)$ of \mathbb{R}^n is called an **orthonormal basis** if b_i and b_j for $i \neq j$ are orthogonal and $\|b_i\| = 1$ for all i , i.e.

$$b_i \bullet b_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} .$$

Theorem 5.14. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a linear map and $A = [F]_B$ the matrix of F for $B = (e_1, \dots, e_n)$. The following statements are equivalent.

- i) F is orthogonal.
- ii) A is orthogonal.
- iii) For all $x, y \in \mathbb{R}^n$ we have $F(x) \bullet F(y) = x \bullet y$.
- iv) A is invertible and $A^{-1} = A^T$.
- v) $(F(e_1), \dots, F(e_n))$ (the columns of A) is an orthonormal basis of \mathbb{R}^n .
- vi) If (b_1, \dots, b_n) is an orthonormal basis of \mathbb{R}^n then $(F(b_1), \dots, F(b_n))$ is also an orthonormal basis.

Corollary 5.15. i) $A \in \mathbb{R}^{n \times n}$ is orthogonal if and only if A^T is orthogonal.

ii) If $A, B \in \mathbb{R}^{n \times n}$ are orthogonal then AB is orthogonal.

iii) If B_1 and B_2 are two orthonormal bases, then the change of basis matrix $S_{B_1}^{B_2}$ is orthogonal.

Definition 5.16. i) An **eigenbasis** of a linear map $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a basis consisting of eigenvectors of F .

ii) Let $U \subset \mathbb{R}^n$ be a subspace. A linear map $F : U \rightarrow U$ is called **symmetric** if we have for all $x, y \in U$

$$x \bullet F(y) = F(x) \bullet y .$$

Theorem 5.17. (Spectral theorem) Let $U \subset \mathbb{R}^n$ be a subspace and $F : U \rightarrow U$ a linear map. Then F is symmetric if and only if there exists an orthonormal eigenbasis of F .

Corollary 5.18. *A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if and only if there exists an orthogonal matrix $S \in \mathbb{R}^{n \times n}$, such that*

$$S^T A S = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix},$$

for some $d_1, \dots, d_n \in \mathbb{R}$.

Lemma 5.19. *Every symmetric linear map $F : U \rightarrow U$ has an eigenvalue.*

6 Linear differential equations

Let $x : \mathbb{R} \rightarrow \mathbb{R}^n$ be a function written as

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

where the entries x_1, \dots, x_n are differentiable functions in $C^1(\mathbb{R}, \mathbb{R})$. By $x'(t) = \frac{d}{dt}x(t)$ we denote

$$x'(t) = \begin{pmatrix} x'_1(t) \\ \vdots \\ x'_n(t) \end{pmatrix}.$$

For a matrix $A \in \mathbb{R}^{n \times n}$ the equation

$$x'(t) = Ax(t)$$

is called a **continuous (linear) dynamical system**.

One dimensional ($n = 1$) continuous dynamical systems have the following solutions:

Proposition 6.1. *Let $a \in \mathbb{R}$. The only solutions to*

$$x'(t) = ax(t)$$

in $C^1(\mathbb{R}, \mathbb{R})$ are given by $x(t) = ce^{at}$ for $c \in \mathbb{R}$.

Recall that the space $C^\infty(\mathbb{R}, \mathbb{R})$, the space of **smooth functions**, denotes the space of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ for which derivatives of all orders exist. This means that for any $n \geq 0$ and $f \in C^\infty(\mathbb{R}, \mathbb{R})$, the n -th derivative $f^{(n)} \in C^\infty(\mathbb{R}, \mathbb{R})$ exists. The space $C^\infty(\mathbb{R}, \mathbb{R})$ is a vector space.

Definition 6.2. *i) A differential operator of order n is a map $T : C^\infty(\mathbb{R}, \mathbb{R}) \rightarrow C^\infty(\mathbb{R}, \mathbb{R})$ of the form*

$$T(f) = a_0f + a_1f' + a_2f^{(2)} + \dots + a_nf^{(n)}$$

for some $a_0, a_1, \dots, a_n \in \mathbb{R}$ with $a_n \neq 0$.

(More precisely this is a "linear differential operator of order n with constant coefficients".)

ii) A **linear differential equation** is an equation of the form $T(f) = g$, where T is a differential operator and $g \in C^\infty(\mathbb{R}, \mathbb{R})$.

iii) A linear differential equation is called **homogeneous** if $g = 0$, i.e. if $T(f) = 0$.

Lemma 6.3. Let $F : V \rightarrow W$ be a linear map between two vector spaces V and W . Assume that $F(v) = w$ for a fixed $v \in V$ and $w \in W$. Then the following two statements are equivalent:

i) $F(x) = w$.

ii) $x = v + u$ for some $u \in \ker(F)$.

Theorem 6.4. Let $T : C^\infty(\mathbb{R}, \mathbb{R}) \rightarrow C^\infty(\mathbb{R}, \mathbb{R})$ be a differential operator of order n . Then we have

$$\dim(\ker(T)) = n.$$

Definition 6.5. Let $T(f) = a_0f + a_1f' + \dots + a_nf^{(n)}$ be a differential operator of order n . The **characteristic polynomial** of T is defined by

$$p_T(x) = \sum_{i=0}^n a_i x^i = a_0 + a_1x + \dots + a_nx^n.$$

In the following, T always denotes a differential operator.

Proposition 6.6. i) The function $e^{\lambda t}$ is an eigenvector of T with eigenvalue $p_T(\lambda)$.

ii) We have $e^{\lambda t} \in \ker(T)$ if and only if $p_T(\lambda) = 0$.

Corollary 6.7. Let T be a differential operator of order n .

i) If $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ are distinct, then $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$ are linearly independent.

ii) If p_T has n distinct zeroes $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ then $(e^{\lambda_1 t}, \dots, e^{\lambda_n t})$ is a basis of $\ker(T)$.

Lemma 6.8. For two differential operators T_1 and T_2 we have $T_1 \circ T_2 = T_2 \circ T_1$.

Theorem 6.9. Let T be a differential operator with characteristic polynomial

$$p_T(x) = (x - \lambda_1)^{m_1} \dots (x - \lambda_r)^{m_r}$$

where $\lambda_1, \dots, \lambda_r \in \mathbb{R}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$.

Then $B = B_1 \cup \dots \cup B_r$ is a basis of $\ker(T)$, where we have for $1 \leq j \leq r$

$$B_j = (e^{\lambda_j t}, te^{\lambda_j t}, \dots, t^{m_j-1}e^{\lambda_j t}).$$

Theorem 6.10. Let T be a differential operator. If $p_T(x)$ contains a factor $((x - a)^2 + b^2)^m$, then

$$\{e^{at} \cos(bt), e^{at} \sin(bt), te^{at} \cos(bt), te^{at} \sin(bt), \dots, t^{m-1}e^{at} \cos(bt), t^{m-1}e^{at} \sin(bt)\}$$

are $2m$ linearly independent elements in $\ker(T)$.

Lemma 6.11. Let $F : U \rightarrow V$ and $G : V \rightarrow W$ be surjective linear maps between vector spaces U, V, W , such that $\ker(F)$ and $\ker(G)$ are finitely generated. Then we have

$$\dim(\ker(G \circ F)) = \dim(\ker(F)) + \dim(\ker(G)).$$