Linear Algebra II - Midterm

Nagoya University, G30 Program

Spring 2022

Instructor: Henrik Bachmann

Total: 36 Points

1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.

i) The set $U = \{ f \in C^\infty(\mathbb{R},\mathbb{R}) \mid f(1)^2 \geq f(1) \}$ is a subspace of $C^\infty(\mathbb{R},\mathbb{R})$.

ii) If V, W are two vector spaces with $\dim(V) = 3$ and $\dim(W) = 2$ then there exists a surjective linear map $F : V \to W$.

iii) For any $A \in \mathbb{R}^{n\times n}$ we have $\det(A) = \det(\mathrm{rref}(A))$.

iv) If (A, B, C, D) is a basis of $\mathbb{R}^{2\times 2}$, then (AA, AB, AC, AD) is also a basis of $\mathbb{R}^{2\times 2}$.

2) (4+4=8 Points) Consider the bases $B = (2x + 1, x + 1)$ and $C = (x, x - 1)$ of $\mathcal{P}_1$ and the linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ with

$$[F]^B_C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $g \in \mathcal{P}_1$ with $g(x) = x$.

i) Determine $F(g)$.

ii) Calculate the determinant of F.

(You do <u>not</u> need to show that B and C are bases)

3) (6 Points) For $n \geq 1$ consider the matrix $A_n = (a_{ij}) \in \mathbb{R}^{n\times n}$ with $a_{ij} = i + j$, i.e.

$$A_n = \begin{pmatrix} 2 & 3 & \dots & n+1 \\ 3 & 4 & \dots & n+2 \\ \vdots & \dots & \ddots & \vdots \\ n+1 & n+2 & \dots & n+n \end{pmatrix}.$$

Determine $\det(A_n)$ for all $n \geq 1$.

4) (3+3+3=9 Points) We define the following elements in $\mathbb{R}^{2\times 2}$

$$m_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; m_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \; m_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \; m_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

and define the following linear map

$$\begin{aligned} G : \mathbb{R}^{2\times 2} &\longrightarrow \mathbb{R}^{2\times 2} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} b & a \\ d & c \end{pmatrix}. \end{aligned}$$

(You do <u>not</u> need to show that G is linear)

i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2\times 2}$.

ii) Show that $N = (G(m_1), G(m_2), G(m_3), G(m_4))$ is a basis of $\mathbb{R}^{2\times 2}$.

iii) Determine $[G]_M$ and $[G]^N_M$.

5) (5 Points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Show by induction that for all $n \geq 0$ we have

$$\sum_{j=0}^{n} F_j^2 = F_n F_{n+1}.$$