Linear Algebra II

Lecture 10: **Spectral theorem & Applications**

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Download these slides at https://www.henrikbachmann.com/la2_2022.html.

There you can also find a link to a Google Colab notebook, which you can use during the lecture

Symmetric linear maps

Definition 5.16

- An **eigenbasis** of a linear map $F : \mathbb{R}^n \to \mathbb{R}^n$ is a basis consisting of eigenvectors of F.
- Let $U \subset \mathbb{R}^n$ be a subspace. A linear map $F : U \to U$ is called **symmetric** if we have for all $x, y \in U$

$$x \bullet F(y) = F(x) \bullet y\,.$$

We saw last lecture:

$$F \text{ is symmetric} \quad \iff \quad [F]_B \text{ is a symmetric matrix for an orthonormal basis } B\,.$$

Spectral theorem

Theorem 5.17 (Spectral Theorem)

Let $U \subset \mathbb{R}^n$ be a subspace and $F : U \to U$ a linear map. Then F is symmetric if and only if there exists an orthonormal eigenbasis of F.

As a special case we have the following:

Corollary 5.18

A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if and only if there exists an orthogonal matrix $S \in \mathbb{R}^{n\times n}$, such that

$$S^T A S = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix},$$

for some $d_1, \ldots, d_n \in \mathbb{R}$.

Proof of the Corollary

The reflection example

In the lecture we considered the matrix

$$A = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$$

and calculated its eigenvalues and eigenvectors. We saw that the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$. The eigenspaces are

$$E_1(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}, \qquad E_{-1}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Question

How can we find an orthogonal matrix S, such that

$$S^T A S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}?$$

Proof of the spectral theorem

Lemma 5.19

Every symmetric linear map $F : U \to U$ has an eigenvalue.
(We proved this in the last tutorial for $U = \mathbb{R}^2$)

To prove the spectral theorem we want to show

$$F : U \to U \text{ is symmetric} \quad \Longleftrightarrow \quad \text{There exists an orthonormal eigenbasis of } F \text{ for } U$$

(easy direction) "$\Leftarrow$": Let B be an orthonormal eigenbasis of F. Then we have

$$[F]_B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Therefore $[F]_B$ is a symmetric matrix, which implies that F is symmetric.

Proof of the spectral theorem

Power iteration

Question

How can you calculate eigenvalues for big matrices?

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Rough idea: If we have a matrix A and a random vector v, then $A^k v$ for large k will give an approximation for an eigenvector of A for its "largest" eigenvalue.

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More precisely we consider the following situation:

- Let $A \in \mathbb{R}^{n \times n}$ be diagonalizable
- Eigenvalues of A: $\lambda_1, \ldots, \lambda_n$ with $\lambda_1 = \cdots = \lambda_d$ and $|\lambda_1| > |\lambda_j|$ for $j = d+1, \ldots, n$. (Here λ_1 is called the **dominant eigenvalue** of A).
- Let $(b_1, \ldots, b_n)$ be a basis of $\mathbb{R}^n$ of eigenvectors b_j with eigenvalue λ_j.
- Suppose we have $v = \sum_{j=1}^{n} \alpha_j b_j$ for $\alpha_j \in \mathbb{R}$, such that $\sum_{j=1}^{d} \alpha_j b_j \neq 0$.

Power iteration

Power iteration

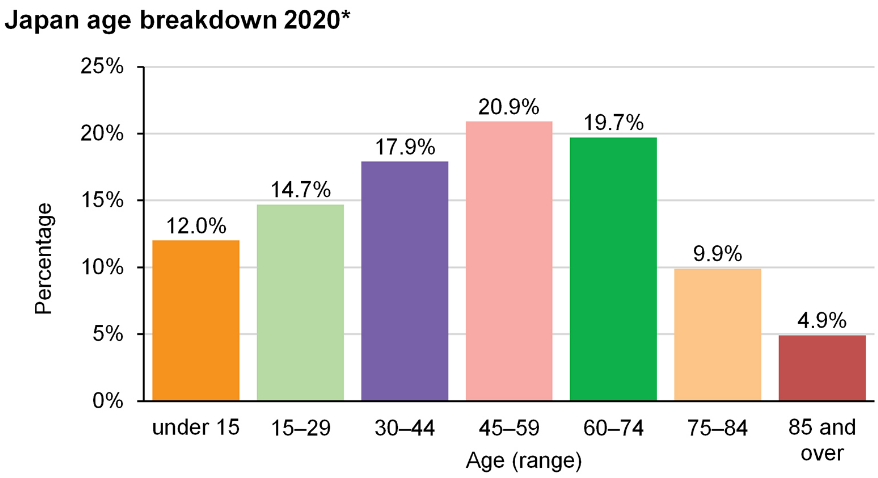
Let $A \in \mathbb{R}^{n \times n}$

- Start with a random vector $v_0 \in \mathbb{R}^n$.
- Define for $k \geq 0$

$$v_{k+1} = \frac{Av_k}{||Av_k||}$$

Then for large k the vector v_k will give an approximation for an eigenvector of A for its dominant eigenvalue.

Population



*Provisional

Population - Example

Consider the following Leslie matrix

$$C = \begin{pmatrix} 0 & 3 & 2 \\ \frac{4}{5} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \end{pmatrix}$$

which models the population behavior of a population with 3 age groups.

Interpretation:

- Three age groups (a_1, a_2, a_3)
- a_1 does not breed and has a survival rate of $80\% (\frac{4}{5})$
- a_2 makes 3 babies (per cycle) and has a survival rate of $40\% (\frac{2}{5})$
- a_3 makes 2 babies (per cycle) and dies out after one generation.