Homework 3: Induction & Determinants I

Deadline: 29th May (23:55 JST), 2022

Exercise 1. (2+2+2 = 6 Points) Use mathematical induction to prove the following statements.

i) For the matrix $M = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$ and all $n \geq 1$ we have

$$M^n = \begin{pmatrix} 2^n & 0 \\ 1-2^n & 1 \end{pmatrix}.$$

ii) For all $n \geq 4$ we have $n! > 2^n$. (Here $n! = 1 \cdot 2 \cdots n$ denotes the factorial.)

iii) Let V be a vector space which is <u>not</u> finitely generated. Then for any $n \geq 1$ there exist vectors $v_1, \ldots, v_n \in V$ which are linearly independent.

Exercise 2. (2+4 = 6 Points) (Geometric interpretation of the determinant)
We define the vectors $v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, u = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^2$.

i) Connect the endpoints of the vectors 0, v, u and $v+u$ to get a parallelogram in $\mathbb{R}^2$. (Make a sketch)

ii) Show that the area of this parallelogram is given by $\det \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$, i.e. the determinant of the matrix which has v and u as columns.

(Remark: This works in general, i.e. if you write two vectors in $\mathbb{R}^2$ into the columns of a matrix $A \in \mathbb{R}^{2\times 2}$ then $|\det(A)|$ gives the area of the parallelogram spanned by them.)

Exercise 3. (4+2 = 6 Points)

i) Show (without using Proposition 4.7) that the determinant is linear in each row, i.e. for any $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ and $1 \leq l \leq n$ show that the map

$$\begin{aligned} F_{A,l} : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \det(A(l;x)) \end{aligned}$$

is linear. Here $A(l;x)$ denotes the matrix A, where the l-th row is replaced by the vector x^T.
(See at the bottom of page 7 in the overview notes)

ii) Assume that A is invertible. What is the kernel of $F_{A,1}$?

Exercise 4. (2+2+2 = 6 Points) We define the matrix

$$A = \begin{pmatrix} -1 & -1 & -2 \\ 2 & 2 & 1 \\ 6 & 2 & 6 \end{pmatrix}$$

and the polynomial $P(\lambda) = \det(A - \lambda I_3)$, where I_3 denotes the 3×3-identity matrix.

i) Calculate $\det(A)$.

ii) Find all solutions $\lambda \in \mathbb{R}$ to $P(\lambda) = 0$.

iii) For each solution λ in ii) find a non-zero vector $v \in \ker(A - \lambda I_3)$ and evaluate Av.
Can you observe a relationship between v, λ and A?