Homework 2: Linear maps and their matrices

Deadline: 15th May (23:55 JST), 2022

Exercise 1. (2+2+2+2+2 = 10 Points) For $n \geq 1$ we define the map $H_n : \mathcal{P}_n \to \mathbb{R}^3$ for a $p \in \mathcal{P}_n$ by

$$H_n(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}.$$

i) Show that H_n is a linear map.

ii) Show that H_2 is an isomorphism.

iii) Calculate the inverse of H_2.

iv) Check if H_1 and H_3 are injective and/or surjective.

v) Determine a basis of $\mathrm{im}(H_1)$.

Exercise 2. (2+3+5 = 10 Points) Consider the basis $B = (1, x, x^2)$ of $\mathcal{P}_2$ and assume we have a linear map $G : \mathcal{P}_2 \to \mathcal{P}_2$ with

$$[G]_B = \begin{pmatrix} 3 & 2 & 0 \\ -1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}. \tag{0.1}$$

i) Determine $G(x^2 + x + 1)$.

ii) Find a (non-zero) polynomial function $f \in \mathcal{P}_2$ with $G(f) = f$.

iii) Find a basis $C = (c_1, c_2, c_3)$ of $\mathcal{P}_2$ such that $[G]_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and determine $G(c_1)$, $G(c_2)$, $G(c_3)$.

Exercise 3. (1+1+2+2 = 6 Points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}. \qquad (n \geq 2)$$

In this exercise we want to prove the following explicit formula

$$F_n = \frac{1}{2^n \sqrt{5}} \left((1+\sqrt{5})^n - (1-\sqrt{5})^n \right). \tag{0.2}$$

For this follow the following steps:

i) Find a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$, such that $F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for $n \geq 1$, where $F^n = \underbrace{F \circ \cdots \circ F}_{n}$.

ii) We define the following two bases of $\mathbb{R}^2$:

$$B_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \quad B_2 = \left(\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} \right).$$

Determine the change-of-basis matrices $S_{B_1}^{B_2}$ and $\left(S_{B_1}^{B_2}\right)^{-1}$.

iii) Calculate $[F]_{B_1}$ and $[F]_{B_2}$.

iv) Calculate $[F]_{B_1}^n$ by using

$$[F]_{B_1} = \left(S_{B_1}^{B_2}\right)^{-1} [F]_{B_2} S_{B_1}^{B_2}$$

and prove (0.2) by using i).

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