Homework 1: Vector spaces

Deadline: 24th April (23:55 JST), 2022

Exercise 1. (2 Points)

i) Try to solve the exercises below and write the solutions down by hand (paper, tablet) or by computer (Latex only. No word!). Write your name, the homework number, and the course name on the first page of your solution. Create **one pdf-file** (for example, by using a scanner app on your phone) and submit it before the deadline ends in NUCT at the Assignment "Homework 1". Use precisely the following format as a filename: "**Familyname_Givenname_LA2_HW1.pdf**". Repeat this for future Homework by replacing HW1 with HW2, HW3, etc.

ii) Read Section 1 of the overview notes and compare the results and definitions with the corresponding results in Linear Algebra I.

Exercise 2. (4+2+2 = 8 Points) Let $V = \mathbb{R}$ be the set of real numbers. Define on V the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$u \oplus v = u + v - 2022\,,$$
$$\lambda \odot v = \lambda(v - 2022) + 2022\,.$$

i) Show that $(V, \oplus, \odot)$ is a vector space. What is the neutral element in V?
(i.e. check that the operations $\oplus$ and $\odot$ satisfy the properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$.)

ii) Determine all subspaces of $(V, \oplus, \odot)$.

iii) Find an isomorphism

$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot)\,.$$

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

Exercise 3. (6 Points) Let $V = \mathbb{R}$ and define on V for $u, v \in V$ the new addition

$$u \boxplus v := \max(u, v)\,,$$

where $\max(u, v)$ denotes the maximum of u and v. For the scalar multiplication we use the usual multiplication $\cdot$ of real numbers. Show that $(V, \boxplus, \cdot)$ is not a vector space. Give a counterexample for all properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$, which are not satisfied in Definition 1.1.

Exercise 4. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\mathcal{P}_3 = \{f \in \mathcal{P} \mid \deg(f) \leq 3\}\,,$$
$$U = \{f \in \mathcal{P}_3 \mid f(1) = f(2) = 0\} \subset \mathcal{P}_3\,.$$

i) Show that U is a subspace of $\mathcal{P}_3$.

ii) Determine a basis $B = (b_1, \ldots, b_n)$ of U.

iii) Determine the coordinate vector $[f]_B$ for the function f given by $f(x) = (x-1)^2(x-2)$.

iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)

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