# Final exam

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頑張って โชคดีในการสอบนะ Chúc may mắn nhé mult succes la examen Semoga berjaya Galingan nyo

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**Exercise 1.** (14 Points) We define the following matrix

$$A = \begin{pmatrix} 3 & -2 & 5 \\ 1 & 0 & 7 \\ 0 & 0 & 2 \end{pmatrix}.$$

i) Compute the determinant of $A$.

ii) Find all eigenvalues of $A$. For each eigenvalue $\lambda$ determine a basis of the eigenspace $E_\lambda(A)$.

iii) Is $A$ diagonalizable and/or invertible and/or orthogonal ? Justify your answers.

**Exercise 2.** (12 Points) Decide if the following statements are true or false. Justify your answer by giving a short explanation.

i) The set $U = \{p \in \mathcal{P}_1 \mid p(1) = p'(1)\}$ is a subspace of $\mathcal{P}_1$.

ii) The set $U = \left\{A \in \mathbb{R}^{2\times 2} \mid A \text{ has eigenvalue } 0\right\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

iii) If $A \in \mathbb{R}^{n\times n}$ is an orthogonal matrix with $A^2 = I_n$, then $A$ is diagonalizable.

iv) For any basis $(b_1, b_2, b_3)$ of $\mathbb{R}^3$ and arbitrary numbers $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ there always exists a matrix $A \in \mathbb{R}^{3\times 3}$ with $Ab_i = \lambda_i b_i$ for $i = 1, 2, 3$.

v) There exists a linear map $F : \mathbb{R}^4 \to \mathbb{R}^2$ with $\dim(\ker(F)) < \dim(\operatorname{im}(F))$.

**Exercise 3.** (8 Points) Give an example of

i) a surjective linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ with $\dim(\ker(F)) = \det(F) + 1$.

ii) a linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ which is not diagonalizable.

iii) a linear map $H : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ which has the eigenvalues $1, 2, 3, 4$.

Justify your examples.

**Exercise 4.** (8 Points) Consider the linear map

$$\begin{aligned} F : C^\infty(\mathbb{R}, \mathbb{R}) &\longrightarrow C^\infty(\mathbb{R}, \mathbb{R}) \\ f &\longmapsto f''. \end{aligned}$$

Show that any $\lambda \in \mathbb{R}$ is an eigenvalue of $F$ and determine all eigenvectors of $F$ with eigenvalue $\lambda$.

**Exercise 5.** (8 Points) Determine all functions $f \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying the equation

$$f(t) = f'''(t) - 3f''(t) + 3f'(t),$$

with $f(0) = 2022$.