Mathematics Tutorial 2b & Linear Algebra II, Spring semester 2021

Question session, 1st May 2021, 14:00 - ...

Today: Ask any questions

Exercise 2. (4+2+2 = 8 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \{ A \in \mathbb{R}^{2\times 2} \mid AM = MA \} .$$

i) Show that $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$ for any $M \in \mathbb{R}^{2\times 2}$.

ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4 .$$

(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)

Exercise 3. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\mathcal{P}_3 = \{ f \in \mathcal{P} \mid \deg(f) \leq 3 \} ,$$
$$U = \{ f \in \mathcal{P}_3 \mid f(-1) = 0 \} \subset \mathcal{P}_3 .$$

i) Show that U is a subspace of $\mathcal{P}_3$.

ii) Determine a basis $B = (b_1, \ldots, b_n)$ of U.

iii) Determine the coordinate vector $[f]_B$ for the function f given by $f(x) = 2(x+1)^3$.

iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)

Exercise 1. (4+2+2 = 8 Points) Let $V = \{x \in \mathbb{R} \mid x > 0\}$ be the set of all positive real numbers. Define on V the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$u \oplus v = uv \qquad \text{(the usual multiplication of real numbers)}$$
$$\lambda \odot v = v^{\lambda} .$$

i) Show that $(V, \oplus, \odot)$ is a vector space.
(i.e. check that the operations $\oplus$ and $\odot$ satisfy the properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$.)

ii) Determine all subspaces of $(V, \oplus, \odot)$.

iii) Find an isomorphism

$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot) .$$

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

1 Vector spaces

Definition 1.1. *A* **(real) vector space** *(linear space) is a set V together with two functions*

Addition

$$\begin{aligned} + : V \times V &\longrightarrow V \\ (u, v) &\longmapsto u + v \end{aligned}$$

Scalar multiplication

$$\begin{aligned} \cdot : \mathbb{R} \times V &\longrightarrow V \\ (\lambda, v) &\longmapsto \lambda \cdot v = \lambda v \end{aligned}$$

satisfying the following properties:

- *Properties of the addition:*

(*A.1*) $\forall u, v, w \in V$*:* $(u + v) + w = u + (v + w)$. *(Associativity)*

(*A.2*) $\forall u, v \in V$*:* $u + v = v + u$. *(Commutativity)*

(*A.3*) $\exists n \in V, \forall u \in V$*:* $n + u = u$. *(Identity/neutral element of addition)*

(*A.4*) $\forall u \in V, \exists v \in V$*:* $u + v = n$. *(Inverse elements of addition)*

- *Compatibility of addition and scalar multiplication:*

(*C.1*) $\forall u, v \in V, \lambda \in \mathbb{R}$*:* $\lambda(u + v) = \lambda u + \lambda v$. *(Distributivity I)*

(*C.2*) $\forall u \in V, \lambda, \mu \in \mathbb{R}$*:* $(\lambda + \mu)u = \lambda u + \mu u$. *(Distributivity II)*

(*C.3*) $\forall u \in V, \lambda, \mu \in \mathbb{R}$*:* $\lambda(\mu u) = (\lambda\mu)u$.

(*C.4*) $\forall u \in V$*:* $1u = u$.

Proposition 1.2. *Let V be a vector space and $u \in V$.*

i) $u + n = u$.

ii) *If $n, \tilde{n} \in V$ both satisfy (A.3) in Definition 1.1, then $n = \tilde{n}$.*
(The Identity element is unique)

iii) *If for a fixed $u \in V$ the elements $v, \tilde{v} \in V$ both satisfy (A.4), i.e. $u + v = u + \tilde{v} = n$, then $v = \tilde{v}$.*
(The inverse of an element u is unique)

iv) $u + (-1)u = 0$.

The identity (also called neutral) element $n \in V$ of a vector space is usually (by abuse of notation) also denoted by 0. Be always aware in the following if 0 means the real number 0 or the identity element of a vector space. (These are two different things!)

Definition 1.3. *Let V be a vector space. A subset $U \subset V$ is a* **subspace** *if*

i) $0 \in U$.

ii) $\forall u, v \in U$*:* $u + v \in U$.

iii) $\forall u \in V, \lambda \in \mathbb{R}$*:* $\lambda u \in U$.

Proposition 1.4. *If $U \subset V$ is a subspace, then U is also a vector space with the operations inherited from V.*

Definition 1.5. *Let V be a vector space and $v_1, \dots, v_n \in V$.*

i) *The* **span** *of the elements $v_1, \dots, v_n$ is given by the set of all their* **linear combinations**, *i.e.*

$$\operatorname{span}\{v_1, \dots, v_n\} = \left\{ \sum_{i=1}^{n} \lambda_i v_i \in V \mid \lambda_1, \dots, \lambda_n \in \mathbb{R} \right\}.$$

ii) *The elements $v_1, \dots, v_n$* **span (or generate) the space** *V if $\operatorname{span}\{v_1, \dots, v_n\} = V$.*

iii) *V is* **finitely generated** *if there exist $v_1, \dots, v_n \in V$ with $\operatorname{span}\{v_1, \dots, v_n\} = V$.*
(i.e. one just needs finitely many elements to generate the space)

iv) *The elements $v_1, \dots, v_n$ are* **linearly independent** *if*

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \implies \lambda_1 = \dots = \lambda_n = 0\,.$$

v) *$B = (v_1, \dots, v_n)$ is a* **basis** *of V if $v_1, \dots, v_n$ are linearly independent and $\operatorname{span}\{v_1, \dots, v_n\} = V$.*

Proposition 1.6. *Let V be a vector space and $v_1, \dots, v_n \in V$. The following statements are equivalent.*

i) *$v_1, \dots, v_n$ are linearly dependent.*

ii) *There exist a $1 \le j \le n$ such that $v_j \in \operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$.*

iii) *There exist a $1 \le j \le n$ such that $\operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\} = \operatorname{span}\{v_1, \dots, v_n\}$.*

Lemma 1.7. *If $v_1, \dots, v_l \in V$ are linearly independent and $V = \operatorname{span}\{w_1, \dots, w_m\}$, then $l \le m$.*

Theorem 1.8. *Let V be a finitely generated vector space. Then we have the following*

i) *V has a (finite) basis.*

ii) *All bases of V have the same number of elements.*

iii) *If $v_1, \dots, v_l \in V$ are linearly independent then there exist $v_{l+1}, \dots, v_n \in V$ such that $(v_1, \dots, v_n)$ is a basis of V.*

iv) *If $V = \operatorname{span}\{w_1, \dots, w_m\}$, then there exist a subset $\{u_1, \dots, u_l\} \subset \{w_1, \dots, w_m\}$, such that $(u_1, \dots, u_l)$ is a basis of V.*

Definition 1.9. *Let V be a finitely generated vector space with basis $(v_1, \dots, v_n)$. Then $\dim(V) = n$ is the* **dimension of V**.

Corollary 1.10. *Let V be a vector space with $\dim(V) = n$ and $v_1, \dots, v_n \in V$. Then the following statements are equivalent.*

i) $v_1, \dots, v_n$ *are linearly independent.*

ii) $V = \operatorname{span}\{v_1, \dots, v_n\}$.

iii) $(v_1, \dots, v_n)$ *is a basis of* V.

Proposition 1.11. *Let* V *be finitely generated and* $U \subset V$ *a subspace. Then* U *is also finitely generated.*

Proposition 1.12. *Let* $B = (v_1, \dots, v_n)$ *be a basis of* V. *Then for all* $u \in V$ *there exist unique* $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, *such that*

$$u = \sum_{i=1}^{n} \lambda_i v_i \,.$$

Definition 1.13. *Let* $B = (v_1, \dots, v_n)$ *be a basis of* V.

i) *The* $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ *in Proposition 1.12 are called the* **coordinates** *of* $u \in V$ *in the basis* B.

ii) *The vector* $[u]_B \in \mathbb{R}^n$ *given by*

$$[u]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is called the **coordinate vector** *of* u *with respect to the basis* B.

2 Linear maps

Definition 2.1. *Let* V, W *be vector spaces. A* **linear map** *is a function* $F : V \to W$ *satisfying*

i) $F(u + v) = F(u) + F(v)$ *for all* $u, v \in V$.

ii) $F(\lambda \cdot u) = \lambda \cdot F(u)$ *for all* $u \in V, \lambda \in \mathbb{R}$.

Definition 2.2. *Let* $F : V \to W$ *be a linear map.*

neutral element in W.

i) *The* **kernel of** F *is given by*

$$\ker(F) = \{u \in V \mid F(u) = 0\} \subset V \,.$$

ii) *The* **image of** F *is given by*

$$\operatorname{im}(F) = \{w \in W \mid \exists u \in V : w = F(u)\} \subset W \,.$$

With the same arguments as in the $\mathbb{R}^n$*-case we see that* $\ker(F)$ *is a subspace of* V *and* $\operatorname{im}(F)$ *is a subspace of* W. *If* $\operatorname{im}(F)$ *is finitely generated, we define the* **rank of** F *by* $\operatorname{rk}(F) = \dim(\operatorname{im}(F))$.

Theorem 2.3 (kernel-image theorem)**.** *Let V be finitely generated and let $F : V \to W$ be a linear map to an arbitrary vector space W. Then*

$$\dim V = \dim(\ker(F)) + \dim(\operatorname{im}(F)) .$$

Definition 2.4. *i) (Recall) A function $f : X \to Y$ is* **invertible** *if there exist a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. f is invertible iff f is bijective, i.e. injective and surjective.*

ii) An invertible linear map $F : V \to W$ is called an **isomorphism**.

iii) Two vector spaces V and W are called **isomorphic** *(Notation: $V \cong W$) if there exists an isomorphism $F : V \to W$.*

Theorem 2.5. *i) A linear map $F : V \to W$ is an isomorphism iff $\ker(F) = \{0\}$ (F is injective) and $\operatorname{im}(F) = W$ (F is surjective).*

ii) Let $F : V \to W$ be an isomorphism and $(b_1, \ldots, b_n)$ a basis of V. Then $(F(b_1), \ldots, F(b_n))$ is a basis of W.

iii) Let V, W be finitely generated and $V \cong W$ then $\dim(V) = \dim(W)$.

iv) Let V, W be finitely generated and $\dim(V) = \dim(W)$. Then for a linear map $F : V \to W$ the following three statements are equivalent

(a) F is an isomorphism.

(b) $\ker(F) = \{0\}$.

(c) $\operatorname{im}(F) = W$.

Proposition 2.6. *Let V be finitely generated with basis $B = (b_1, \ldots, b_n)$, i.e. $\dim(V) = n$. Then the* **coordinate map**

$$\begin{aligned} c_B : \mathbb{R}^n &\longrightarrow V , \\ \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} &\longmapsto \sum_{i=1}^{n} \lambda_i b_i \end{aligned}$$

is an isomorphism. The inverse is given by $c_B^{-1}(u) = [u]_B$ for $u \in V$.

Corollary 2.7. *Let V, W be finitely generated. Then the following two statements are equivalent*

i) $V \cong W$.

ii) $\dim(V) = \dim(W)$.

Homework 1: Vector spaces

Deadline: 2nd May (23:55 JST), 2021

Exercise 1. (4+2+2 = 8 Points) Let $V = \{x \in \mathbb{R} \mid x > 0\}$ be the set of all positive real numbers. Define on V the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$u \oplus v = uv \qquad \text{(the usual multiplication of real numbers)}$$
$$\lambda \odot v = v^\lambda \,.$$

i) Show that $(V, \oplus, \odot)$ is a vector space.
(i.e. check that the operations $\oplus$ and $\odot$ satisfy the properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$.)

ii) Determine all subspaces of $(V, \oplus, \odot)$.

ii*) Determine all subspaces of $(\mathbb{R}^1, +, \cdot)$.
$\{0\}$, $\mathbb{R}$

$\{0, 1\}$
↑
not subspace.

iii) Find an isomorphism

$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot) \,.$$

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

Exercise 2. (4+2+2 = 8 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \left\{A \in \mathbb{R}^{2\times 2} \mid AM = MA\right\} \,.$$

$I_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A = I_2$ works

i) Show that $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$ for any $M \in \mathbb{R}^{2\times 2}$.

ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4 \,.$$

$M = I_2$: $C(I_2) = \mathbb{R}^{2\times 2}$

(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)

Exercise 3. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\mathcal{P}_3 = \{f \in \mathcal{P} \mid \deg(f) \leq 3\} \,,$$
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i) Show that U is a subspace of $\mathcal{P}_3$.

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iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)

For Exercise 1)

$$u \oplus v = uv$$

$$\lambda \odot v = v^{\lambda}.$$

(A.1) $\forall u,v,w \in \{x \in \mathbb{R} \mid x>0\}$

$(u \oplus v) \oplus w \quad u \oplus (v \oplus w)$

$\parallel$ Definition of $\oplus$ $\parallel$

$(u \cdot v) \cdot w \quad = \quad u \cdot (v \cdot w)$

Know from school

Definition 1.1. *A* **(real) vector space** *(linear space) is a set V together with two functions*

Addition	*Scalar multiplication*
$+_V : V \times V \longrightarrow V$	$\cdot : \mathbb{R} \times V \longrightarrow V$
$(u,v) \longmapsto u+v$	$(\lambda, v) \longmapsto \lambda \cdot v = \lambda v$

satisfying the following properties:

- *Properties of the addition:*

$(A.1)$ $\forall u,v,w \in V$: $(u \oplus v) \oplus w = u \oplus (v \oplus w)$. *(Associativity)*

$(A.2)$ $\forall u,v \in V$: $u \oplus v = v \oplus u$. *(Commutativity)*

def $u \cdot v = v \cdot u$

$(A.3)$ $\exists n \in V$, $\forall u \in V$: $n \oplus u = u$. *(Identity/neutral element of addition)*

$n = 1$? $1 \in V$

$(A.4)$ $\forall u \in V$, $\exists v \in V$: $u \oplus v =$ *(Inverse elements of addition)*

- *Compatibility of addition and scalar multiplication:*

$(C.1)$ $\forall u,v \in V$, $\lambda \in \mathbb{R}$: $\lambda(u \oplus v) = \lambda u + \lambda v$. *(Distributivity I)*

$(C.2)$ $\forall u \in V$, $\lambda, \mu \in \mathbb{R}$: $(\lambda + \mu)u = \lambda u + \mu u$. *(Distributivity II)*

$\parallel$

$(\lambda+\mu) \odot u = (\lambda \odot u) \oplus (\mu \odot u)$

$\parallel \qquad \parallel$

$u^{\lambda+\mu} \overset{?}{=} u^{\lambda} \cdot u^{\mu}$

$(C.3)$ $\forall u \in V$, $\lambda, \mu \in \mathbb{R}$: $\lambda(\mu u) = (\lambda\mu)u$.

$(C.4)$ $\forall u \in V$: $1u = u$.

$1 \in \mathbb{R}$

Exercise 1. (4+2+2 = 8 Points) Let $V = \{x \in \mathbb{R} \mid x > 0\}$ be the set of all positive real numbers. Define on V the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

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ii) Determine all subspaces of $(V, \oplus, \odot)$.

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Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

Exercise 2. (4+2+2 = 8 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \{A \in \mathbb{R}^{2\times 2} \mid AM = MA\} .$$

i) Show that $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$ for any $M \in \mathbb{R}^{2\times 2}$.

ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4 .$$

(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)

i) One way: Check the conditions:

Definition 1.3. *Let V be a vector space. A subset $U \subset V$ is a* **subspace** *if*

i) $0 \in U$. ✓

ii) $\forall u, v \in U$: $u + v \in U$.

iii) $\forall u \in V, \lambda \in \mathbb{R}$: $\lambda u \in U$.

$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $V = \mathbb{R}^{2\times 2}$

$U = C(M)$

For ii): If $A \in C(M)$ and $B \in C(M)$ then we want to show that $A+B \in C(M)$

$AM = MA$ $BM = MB$

Check: $(A+B)M = M(A+B)$

Second way: show that $C(M)$ is the kernel or image of a linear map.

i.e. Find a lin. map $F: \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$

$A \longmapsto AM - MA$

s.th. $C(M) = \{A \in \mathbb{R}^{2\times 2} \mid F(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\}$

If F is linear map then $C(M) = \ker(F)$

$\Rightarrow$ $C(M)$ is a subspace.

ii) ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

Want to find all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$AS = SA$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

Wrong: $b = -a$
$c = 2a$
$d = 10a$

$\begin{pmatrix} 1 & -1 \\ 2 & 10 \end{pmatrix}$ is a basis of $C(S)$

$C(S) = \text{span}\left\{\begin{pmatrix} 1 & -1 \\ 2 & 10 \end{pmatrix}\right\}$

iii) iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4\,.$$

(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)

$\dim \mathbb{R}^{2\times 2} = 4$

Basis of $\mathbb{R}^{2\times 2}$

$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$

If $\dim(C(M)) = 0$

$\Updownarrow$

$C(M) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$

Not possible because $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is always (i.e. for any M) in $C(M)$.

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad \overset{I_2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} M = M$ $\qquad I_2 M = M I_2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Exercise 3. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\mathcal{P}_3 = \{f \in \mathcal{P} \mid \deg(f) \leq 3\} \ni f \qquad f(x) = ax^3 + bx^2 + cx + d$$
$$U = \{f \in \mathcal{P}_3 \mid f(-1) = 0\} \subset \mathcal{P}_3 . \qquad a, b, c, d \in \mathbb{R}$$

i) Show that U is a subspace of $\mathcal{P}_3$.

ii) Determine a basis $B = (b_1, \ldots, b_n)$ of U.

iii) Determine the coordinate vector $[f]_B$ for the function f given by $f(x) = 2(x+1)^3$.

iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)

Definition 1.3. *Let V be a vector space. A subset $U \subset V$ is a* **subspace** *if*

i) $\overset{n}{0} \in U$. ✓ $\quad n(x) = 0x^3 + 0x^2 + 0x + 0$

ii) $\forall u, v \in U$: $u + v \in U$. $\quad u(-1) = 0$, $v(-1) = 0$, $((u+v)(-1) = \ldots$, $(u+v)(x) = u(x) + v(x)$

iii) $\forall u \in V, \lambda \in \mathbb{R}$: $\lambda u \in U$. $\quad (\lambda u)(x) = \lambda u(x)$

$V = \mathcal{P}_3$

$\dim \mathcal{P}_n = n+1$

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ii) for the case $W = \{f \in \mathcal{P}_3 \mid f(1) = 0\}$.

$f(x) = ax^3 + bx^2 + cx + d$
$a, b, c, d \in \mathbb{R}$

$f(1) = a + b + c + d = 0$

$a = -b - c - d$
b, c, d are arbitrary.

$B = (b_1, b_2, b_3)$

$b = 1, c = 0, d = 0 \quad a = -1 \quad b_1(x) = -x^3 + x^2$

$b = 0, c = 1, d = 0, \quad a = -1 \quad b_2(x) = -x^3 + x$

$b = 0, c = 0, d = 1, \quad a = -1 \quad b_3(x) = -x^3 + 1$

$W = \operatorname{span}\{b_1, b_2, b_3\}$

Check: b_1, b_2, b_3 lin. indep

$\Rightarrow B = (b_1, b_2, b_3)$ basis.

$$\lambda_1 b_1(x) + \lambda_2 b_2(x) + \lambda_3 b_3(x) = n(x)$$

$$\lambda_1(-x^3+x^2) + \lambda_2(-x^3+x) + \lambda_3(-x^3+1) = 0x^3 + 0x^2 + 0x + 0$$

$$(-\lambda_1-\lambda_2-\lambda_3)x^3 + \lambda_1 x^2 + \lambda_2 x + \lambda_3$$

$$g(x) = x-1 \qquad g \in W$$

iii) Calculate $[g]_B = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$ $\quad g = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3$

$b_1(x) = -x^3 + x^2$

$b_2(x) = -x^3 + x$

$b_3(x) = -x^3 + 1$

Want to solve

$$\underbrace{0x^3+0x^2+x-1}_{g(x)} = \lambda_1 \overbrace{(-x^3+x^2)}^{b_1(x)} + \lambda_2 \overbrace{(-x^3+x)}^{b_2(x)} + \lambda_3 \overbrace{(-x^3+1)}^{b_3(x)}$$

$$= (-\lambda_1-\lambda_2-\lambda_3)x^3 + \lambda_1 x^2 + \lambda_2 x + \lambda_3$$

$$\Rightarrow \quad \lambda_1 = 0, \ \lambda_2 = 1, \ \lambda_3 = -1$$

$$\Rightarrow \quad [g]_B = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \qquad [b_2]_B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

iv): Want to find b_4 s.th. (b_1, b_2, b_3, b_4) is a basis of P_3.

Check: $b_4(x) = x^3$