Exercise 2. (4+2=6 Points) Define for $A \in \mathbb{R}^{m \times n}$ the following set

\[ \mathcal{U}(A) = \{ X \in \mathbb{R}^{n \times m} \mid AX = MA \}. \]

i) Show that $\mathcal{U}(A)$ is a subspace of $\mathbb{R}^{m \times n}$ for any $A \in \mathbb{R}^{m \times n}$.

ii) For $F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ determine a basis of $\mathcal{U}(F)$.

iii) Show that for all $M \in \mathbb{R}^{m \times n}$ we have

\[ 2 \leq \text{det}(UMU^T) \leq 4, \]

i.e., show that these entries in matrix $M$ make that $\mathcal{U}(M)$ has determinant 2 or 4.

Exercise 3. (2+2=4 Points) Let $P_n$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets:

\[ P_n = \{ f \in \mathbb{R} \mid \deg(f) \leq n \}, \]

\[ F = \{ f \in P_1 \mid f(0) = 0 \}. \]

i) Show that $P_n$ is a subspace of $\mathbb{R}[x]$.

ii) Determine a basis $\{ P_1, \ldots, P_n \}$ of $P_n$.

iii) Determine the coordinate vector $[f]_{\mathcal{B}}$ for the function $f(x) = x^3 + 2x + 1$.

iv) Show that $F$ is a subspace of $\mathbb{R}[x]$.

v) Find a basis of $F$ which contains all the basis elements of any basis $\{ P_1 \}$ of $P_1$.

Exercise 1. (4+3=7 Points) Let $V = \{ x \in \mathbb{R} \mid x > 0 \}$ be the set of all positive real numbers. Define on $V$ the addition $\oplus$ and the scalar multiplication $\circ$ by $a \oplus v = av$ (the usual multiplication of real numbers)

\[ a \circ v = v^a. \]

i) Show that $(V, \oplus, \circ)$ is a vector space.

ii) Check that the operations $\oplus$ and $\circ$ satisfy the properties (A.1) – (A.4) and (C.1) – (C.4).

iii) Determine all subspaces of $(V, \oplus, \circ)$.

iv) Find an isomorphism $F : (\mathbb{R}, +, \cdot) \rightarrow (V, \oplus, \circ)$.

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}$ with the usual addition and multiplication of real numbers.
### 1 Vector spaces

**Definition 1.1.** A (real) vector space (linear space) is a set $V$ together with two functions

\[
\begin{align*}
\text{Addition} & \quad + : V \times V \rightarrow V \\
& \quad (u, v) \mapsto u + v \\
\text{Scalar multiplication} & \quad \cdot : \mathbb{R} \times V \rightarrow V \\
& \quad (\lambda, v) \mapsto \lambda \cdot v = \lambda v
\end{align*}
\]

satisfying the following properties:

• **Properties of the addition:**
  
  (A.1) $\forall u, v, w \in V: (u + v) + w = u + (v + w)$. *(Associativity)*
  
  (A.2) $\forall u, v \in V: u + v = v + u$. *(Commutativity)*
  
  (A.3) $\exists n \in V, \forall u \in V: n + u = u$. *(Identity/neutral element of addition)*
  
  (A.4) $\forall u \in V, \exists v \in V: u + v = n$. *(Inverse elements of addition)*

• **Compatibility of addition and scalar multiplication:**
  
  (C.1) $\forall u, v \in V, \lambda \in \mathbb{R}: \lambda(u + v) = \lambda u + \lambda v$. *(Distributivity I)*
  
  (C.2) $\forall u \in V, \lambda, \mu \in \mathbb{R}: (\lambda + \mu)u = \lambda u + \mu u$. *(Distributivity II)*
  
  (C.3) $\forall u \in V, \lambda, \mu \in \mathbb{R}: \lambda(\mu u) = (\lambda \mu)u$.
  
  (C.4) $\forall u \in V: 1u = u$.

**Proposition 1.2.** Let $V$ be a vector space and $u \in V$.

i) $u + n = u$.

ii) If $n, \tilde{n} \in V$ both satisfy (A.3) in Definition 1.1, then $n = \tilde{n}$. *(The identity element is unique)*

iii) If for a fixed $u \in V$ the elements $v, \tilde{v} \in V$ both satisfy (A.4), i.e. $u + v = u + \tilde{v} = n$, then $v = \tilde{v}$. *(The inverse of an element $u$ is unique)*

iv) $u + (-1)u = 0$.

The identity (also called neutral) element $n \in V$ of a vector space is usually (by abuse of notation) also denoted by 0. Be always aware in the following if 0 means the real number 0 or the identity element of a vector space. (These are two different things!)

**Definition 1.3.** Let $V$ be a vector space. A subset $U \subset V$ is a subspace if

i) $0 \in U$.

ii) $\forall u, v \in U: u + v \in U$.

iii) $\forall u \in V, \lambda \in \mathbb{R}: \lambda u \in U$. 

---

Version 1 (April 12, 2021)
Proposition 1.4. If $U \subset V$ is a subspace, then $U$ is also a vector space with the operations inherited from $V$.

Definition 1.5. Let $V$ be a vector space and $v_1, \ldots, v_n \in V$.

i) The span of the elements $v_1, \ldots, v_n$ is given by the set of all their linear combinations, i.e.
\[
\text{span}\{v_1, \ldots, v_n\} = \left\{ \sum_{i=1}^{n} \lambda_i v_i \in V \mid \lambda_1, \ldots, \lambda_n \in \mathbb{R} \right\}.
\]

ii) The elements $v_1, \ldots, v_n$ span (or generate) the space $V$ if $\text{span}\{v_1, \ldots, v_n\} = V$.

iii) $V$ is finitely generated if there exist $v_1, \ldots, v_n \in V$ with $\text{span}\{v_1, \ldots, v_n\} = V$.

i.e. one just needs finitely many elements to generate the space

iv) The elements $v_1, \ldots, v_n$ are linearly independent if
\[
\lambda_1 v_1 + \cdots + \lambda_n v_n = 0 \implies \lambda_1 = \ldots = \lambda_n = 0.
\]

v) $B = (v_1, \ldots, v_n)$ is a basis of $V$ if $v_1, \ldots, v_n$ are linearly independent and $\text{span}\{v_1, \ldots, v_n\} = V$.

Proposition 1.6. Let $V$ be a vector space and $v_1, \ldots, v_n \in V$. The following statements are equivalent.

i) $v_1, \ldots, v_n$ are linearly dependent.

ii) There exist a $1 \leq j \leq n$ such that $v_j \in \text{span}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$.

iii) There exist a $1 \leq j \leq n$ such that $\text{span}\{v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\} = \text{span}\{v_1, \ldots, v_n\}$.

Lemma 1.7. If $v_1, \ldots, v_l \in V$ are linearly independent and $V = \text{span}\{w_1, \ldots, w_m\}$, then $l \leq m$.

Theorem 1.8. Let $V$ be a finitely generated vector space. Then we have the following

i) $V$ has a (finite) basis.

ii) All bases of $V$ have the same number of elements.

iii) If $v_1, \ldots, v_l \in V$ are linearly independent then there exist $v_{l+1}, \ldots, v_n \in V$ such that $(v_1, \ldots, v_n)$ is a basis of $V$.

iv) If $V = \text{span}\{w_1, \ldots, w_m\}$, then there exist a subset $(u_1, \ldots, u_l) \subset \{w_1, \ldots, w_m\}$, such that $(u_1, \ldots, u_l)$ is a basis of $V$.

Definition 1.9. Let $V$ be a finitely generated vector space with basis $(v_1, \ldots, v_n)$. Then $\text{dim}(V) = n$ is the dimension of $V$.

Corollary 1.10. Let $V$ be a vector space with $\text{dim}(V) = n$ and $v_1, \ldots, v_n \in V$. Then the following statements are equivalent.
i) \(v_1, \ldots, v_n\) are linearly independent.

ii) \(V = \text{span}\{v_1, \ldots, v_n\}\).

iii) \((v_1, \ldots, v_n)\) is a basis of \(V\).

**Proposition 1.11.** Let \(V\) be finitely generated and \(U \subset V\) a subspace. Then \(U\) is also finitely generated.

**Proposition 1.12.** Let \(B = (v_1, \ldots, v_n)\) be a basis of \(V\). Then for all \(u \in V\) there exist unique \(\lambda_1, \ldots, \lambda_n \in \mathbb{R}\), such that

\[
    u = \sum_{i=1}^{n} \lambda_i v_i.
\]

**Definition 1.13.** Let \(B = (v_1, \ldots, v_n)\) be a basis of \(V\).

i) The \(\lambda_1, \ldots, \lambda_n \in \mathbb{R}\) in Proposition 1.12 are called the *coordinates* of \(u \in V\) in the basis \(B\).

ii) The vector \([u]_B \in \mathbb{R}^n\) given by

\[
    [u]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}
\]

is called the *coordinate vector* of \(u\) with respect to the basis \(B\).

### 2 Linear maps

**Definition 2.1.** Let \(V, W\) be vector spaces. A *linear map* is a function \(F : V \to W\) satisfying

i) \(F(u + v) = F(u) + F(v)\) for all \(u, v \in V\).

ii) \(F(\lambda \cdot u) = \lambda \cdot F(u)\) for all \(u \in V, \lambda \in \mathbb{R}\).

**Definition 2.2.** Let \(F : V \to W\) be a linear map.

i) The *kernel* of \(F\) is given by

\[
    \ker(F) = \{u \in V \mid F(u) = 0\} \subset V.
\]

ii) The *image* of \(F\) is given by

\[
    \text{im}(F) = \{w \in W \mid \exists u \in V : w = F(u)\} \subset W.
\]

With the same arguments as in the \(\mathbb{R}^n\)-case we see that \(\ker(F)\) is a subspace of \(V\) and \(\text{im}(F)\) is a subspace of \(W\). If \(\text{im}(F)\) is finitely generated, we define the *rank of \(F\)* by \(\text{rk}(F) = \dim(\text{im}(F))\).
**Theorem 2.3** (kernel-image theorem). Let $V$ be finitely generated and let $F: V \to W$ be a linear map to an arbitrary vector space $W$. Then

$$\dim V = \dim(\ker(F)) + \dim(\text{im}(F)).$$

**Definition 2.4.**

i) (Recall) A function $f: X \to Y$ is invertible if there exist a function $g: Y \to X$ such that $f \circ g = \text{id}_Y$ and $g \circ f = \text{id}_X$. $f$ is invertible iff $f$ is bijective, i.e. injective and surjective.

ii) An invertible linear map $F: V \to W$ is called an isomorphism.

iii) Two vector spaces $V$ and $W$ are called isomorphic (Notation: $V \cong W$) if there exists an isomorphism $F: V \to W$.

**Theorem 2.5.**

i) A linear map $F: V \to W$ is an isomorphism iff $\ker(F) = \{0\}$ ($F$ is injective) and $\text{im}(F) = W$ ($F$ is surjective).

ii) Let $F: V \to W$ be an isomorphism and $(b_1, \ldots, b_n)$ a basis of $V$. Then $(F(b_1), \ldots, F(b_n))$ is a basis of $W$.

iii) Let $V, W$ be finitely generated and $V \cong W$ then $\dim(V) = \dim(W)$.

iv) Let $V, W$ be finitely generated and $\dim(V) = \dim(W)$. Then for a linear map $F: V \to W$ the following three statements are equivalent

(a) $F$ is an isomorphism.

(b) $\ker(F) = \{0\}$.

(c) $\text{im}(F) = W$.

**Proposition 2.6.** Let $V$ be finitely generated with basis $B = (b_1, \ldots, b_n)$, i.e. $\dim(V) = n$. Then the coordinate map

$$c_B: \mathbb{R}^n \to V,$$

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \mapsto \sum_{i=1}^{n} \lambda_i b_i$$

is an isomorphism. The inverse is given by $c_B^{-1}(u) = [u]_B$ for $u \in V$.

**Corollary 2.7.** Let $V, W$ be finitely generated. Then the following two statements are equivalent

i) $V \cong W$.

ii) $\dim(V) = \dim(W)$. 
Homework 1: Vector spaces

Exercise 1. (4+2+2 = 8 Points) Let $V = \{x \in \mathbb{R} \mid x > 0\}$ be the set of all positive real numbers. Define on $V$ the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$u \oplus v = uv \quad \text{(the usual multiplication of real numbers)}$$

$$\lambda \odot v = v^\lambda.$$

i) Show that $(V, \oplus, \odot)$ is a vector space.

(ii) Determine all subspaces of $(V, \oplus, \odot)$.

iii) Find an isomorphism \( F : (\mathbb{R}, +, \cdot) \to (V, \oplus, \odot). \)

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

Exercise 2. (4+2+2 = 8 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \{A \in \mathbb{R}^{2\times 2} \mid AM = MA\}.$$

i) Show that $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$ for any $M \in \mathbb{R}^{2\times 2}$.

ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4.$$

(i.e. show that there exists no matrix $M$, such that $C(M)$ has dimension 0 or 1.)

Exercise 3. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\mathcal{P}_3 = \{f \in \mathcal{P} \mid \deg(f) \leq 3\},$$

$$U = \{f \in \mathcal{P}_3 \mid f(-1) = 0\} \subset \mathcal{P}_3.$$

i) Show that $U$ is a subspace of $\mathcal{P}_3$.

ii) Determine a basis $B = (b_1, \ldots, b_n)$ of $U$.

iii) Determine the coordinate vector $[f]_B$ for the function $f$ given by $f(x) = 2(x + 1)^3$.

iv) Extend the basis $B$ to a basis $\tilde{B}$ of $\mathcal{P}_3$.

(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis $B$ of $U$)
For Exercise 1)

\[ u \oplus v = uv \]

\[ \lambda \circ v = v^\lambda. \]

\((A.1)\) \( \forall u, v, w \in \mathbb{R} | x > 0 \) \hspace{1cm} \((u \oplus v) \oplus w = (u \oplus v) \oplus w \) \hspace{1cm} \((u \cdot v) \cdot w = u \cdot (v \cdot w) \) 

\[ \text{Definition} \hspace{1cm} u \oplus v = uv \]

\[ \text{Definition} \hspace{1cm} u \cdot v = uv \]

\[ \text{Definition} \hspace{1cm} u \circ v = u^v \]

\[ \text{Definition} \hspace{1cm} u \oplus v = uv \]

\[ \text{Definition} \hspace{1cm} u \cdot v = uv \]

\[ \text{Definition} \hspace{1cm} u \circ v = u^v \]

Definition 1.1. A (real) vector space (linear space) is a set \( V \) together with two functions

\[ \begin{align*}
&\text{Addition} \quad V \times V \rightarrow V \\
&(\mu \cdot v) \rightarrow \mu \cdot v \quad \text{(Scalar multiplication)}
\end{align*} \]

satisfying the following properties:

- Properties of the addition:
  \( (A.1) \) \( \forall u, v, w \in V \) \( u \oplus (v \oplus w) = (u \oplus v) \oplus w \) (Associativity)
  \( (A.2) \) \( \forall u, v \in V \) \( u \oplus v = v \oplus u \) (Commutativity)
  \( (A.3) \) \( \exists 0 \in V \) \( \forall u \in V \) \( u \oplus 0 = u \) (Identity/neutral element of addition)
  \( (A.4) \) \( \forall u \in V \) \( \exists u^{-1} \in V \) \( u \oplus u^{-1} = 0 \) (Inverse elements of addition)

- Compatibility of addition and scalar multiplication:
  \( (C.1) \) \( \forall u, v \in V, \lambda \in \mathbb{R} \) \( \lambda (u \oplus v) = \lambda u \oplus \lambda v \) (Distributivity I)
  \( (C.2) \) \( \forall u \in V, \lambda, \mu \in \mathbb{R} \) \( (\lambda \mu) u = \lambda (\mu u) \) (Distributivity II)
  \( (C.3) \) \( \forall u \in V, \lambda, \mu \in \mathbb{R} \) \( \lambda (\mu u) = (\lambda \mu) u \) (Associativity of scalar multiplication)
  \( (C.4) \) \( \forall u \in V \) \( \lambda \cdot u = u \lambda \) (Conjugate symmetry of scalar multiplication)

Exercise 1. \( (4+2=8 \text{ Points}) \) Let \( V = (x \in \mathbb{R} | x > 0) \) be the set of all positive real numbers. Define on \( V \) the addition \( \oplus \) and the scalar multiplication \( \circ \) for \( u, v \in V \) and \( \lambda \in \mathbb{R} \) by

\[ w = u \oplus v \]

(definition of \( u \oplus v \))

\[ \lambda \circ u = \lambda u \]

(definition of \( \lambda \circ u \))

1) Show that \( (V, \oplus, \circ) \) is a vector space.
   (i.e., check that the operations \( \oplus \) and \( \circ \) satisfy the properties \( (A.1) \) - \( (A.4) \) and \( (C.1) \) - \( (C.6) \).)

2) Determine all subspaces of \( (V, \oplus, \circ) \).

3) Find an isomorphism \( F : (\mathbb{R}, +, \cdot) \rightarrow (V, \oplus, \circ) \).

Here \( (\mathbb{R}, +, \cdot) \) denotes the vector space \( \mathbb{R}^1 \) with the usual addition and multiplication of real numbers.
Exercise 2. (4+2+2 = 8 Points) Define for $M \in \mathbb{R}^{2 \times 2}$ the following set

$$C(M) = \{ A \in \mathbb{R}^{2 \times 2} \mid AM = MA \}.$$ 

i) Show that $C(M)$ is a subspace of $\mathbb{R}^{2 \times 2}$ for any $M \in \mathbb{R}^{2 \times 2}$.

ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

iii) Show that for all $M \in \mathbb{R}^{2 \times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4.$$ 

(i.e. show that there exists no matrix $M$, such that $C(M)$ has dimension 0 or 1.)

i) One way: Check the conditions:

Definition 1.3. Let $V$ be a vector space. A subset $U \subset V$ is a subspace if

1. $0 \in U$. 
2. $\forall u, v \in U: u + v \in U$. 
3. $\forall u \in V, \lambda \in \mathbb{R}: \lambda u \in U$.

For ii): If $A \in C(M)$ and $B \in C(M)$ then we want to show that $A + B \in C(M)$.

Check: $(A + B)M = M(A + B)$

Second way: Show that $C(M)$ is the kernel or image of a linear map.

i.e. Find a linear map $F: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$

$A \mapsto AM - MA$

s.t. $C(M) = \{ A \in \mathbb{R}^{2 \times 2} \mid F(A) = (0, 0) \}$

If $F$ is linear map then $C(M) = \ker(F)$

$\Rightarrow$ $C(M)$ is a subspace.
(i) ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

Want to find all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$AS = SA$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Wrong: $b = -a$
$c = 2a$
$d = 10a$

$\begin{pmatrix} 1 & -1 \\ 2 & 10 \end{pmatrix}$ is a basis of $C(S)$

$C(S) = \text{span}\{\begin{pmatrix} 1 & -1 \\ 2 & 10 \end{pmatrix}\}$

(iii) Show that for all $M \in \mathbb{R}^{2 \times 2}$ we have

$2 \leq \text{dim}(C(M)) \leq 4$.

(i.e. show that there exists no matrix $M$, such that $C(M)$ has dimension 0 or 1.)

If $\text{dim}(C(M)) = 0$

\[C(M) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}\]

Not possible because $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is always (i.e. for any $M$) in $C(M)$. 

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} M = M$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$I_2 M = M I_2$
Exercise 3. \((2+2+2=8\text{ Points})\) Let \(P\) denote the set of all polynomial functions from \(\mathbb{R}\) to \(\mathbb{R}\). Define the following subsets

\[
P_3 = \{f \in P \mid \deg(f) \leq 3\} \quad f(x) = a x^3 + b x^2 + c x + d
\]

\[U = \{f \in P_3 \mid f(-1) = 0\} \subset P_3.
\]

i) Show that \(U\) is a subspace of \(P_3\).

ii) Determine a basis \(B = (b_1, \ldots, b_n)\) of \(U\).

iii) Determine the coordinate vector \([f]_B\) for the function \(f\) given by \(f(x) = 2(x+1)^3\).

iv) Extend the basis \(B\) to a basis \(\tilde{B}\) of \(P_3\).

(i.e. find a basis of \(P_3\), which contains all the basis elements of your basis \(B\) of \(U\))

---

**Definition 1.3.** Let \(V\) be a vector space. A subset \(U \subset V\) is a subspace if

i) \(0 \in U\).

ii) \(\forall u, v \in U: u + v \in U\).

iii) \(\forall u \in U, \lambda \in \mathbb{R}: \lambda u \in U\).

\[V = P_3\]

\[\lambda u(x) = \lambda u(x)\]

\[(u+v)(x) = u(x) + v(x)\]

---

**ii) for the case \(W = \{f \in P_3 \mid f(1) = 0\}\)**

\[f(x) = a x^3 + b x^2 + c x + d \quad a, b, c, d \in \mathbb{R}\]

\[f(1) = a + b + c + d = 0\]

\[
\begin{align*}
b &= 1, c = 0, d = 0 \quad a = -1 & b_1(x) &= -x^3 + x^2 \\
b &= 0, c = 1, d = 0, a = -1 & b_2(x) &= -x^3 + x \\
b &= 0, c = 0, d = 1, a = -1 & b_3(x) &= -x^3 + 1
\end{align*}
\]

\[W = \text{Span}\{b_1, b_2, b_3\}\]

Check: \(b_1, b_2, b_3\) lin. ind. \(\Rightarrow\) \(B = (b_1, b_2, b_3)\) basis.
\[ \lambda_1 \hat{b}_1(x) + \lambda_2 \hat{b}_2(x) + \lambda_3 \hat{b}_3(x) = n(x) \]

\[ \lambda_1 \left(-x^2+x\right) + \lambda_2 \left(-x^3+x\right) + \lambda_3 \left(-x^3+1\right) \]

\[ (-\lambda_1-\lambda_2-\lambda_3)x^3 + \lambda_1 x^2 + \lambda_2 x + \lambda_3 \]

\[ g(x) = x - 1 \quad g \in W \]

iii) \[ \text{Calculate } \begin{bmatrix} g \end{bmatrix}_B = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \quad g = \lambda_1 \hat{b}_1 + \lambda_2 \hat{b}_2 + \lambda_3 \hat{b}_3 \]

Want to solve \( \hat{b}_1(x) \hat{b}_2(x) \hat{b}_3(x) \)

\[ 0x^3+0x^2+x-1 = \lambda_1 \left(-x^2+x\right) + \lambda_2 \left(-x^3+x\right) + \lambda_3 \left(-x^3+1\right) \]

\[ g(x) = (-\lambda_1-\lambda_2-\lambda_3)x^3 + \lambda_1 x^2 + \lambda_2 x + \lambda_3 \]

\[ \Rightarrow \lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = -1 \]

\[ \Rightarrow \begin{bmatrix} g \end{bmatrix}_B = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \begin{bmatrix} b_1 \end{bmatrix}_B = \begin{pmatrix} 0 \end{pmatrix} \]

iv) Want to find \( b_4 \) s.t. \( (b_1, b_2, b_3, b_4) \)

is a basis of \( P_3 \).

Check: \( b_4(x) = x^3 \)