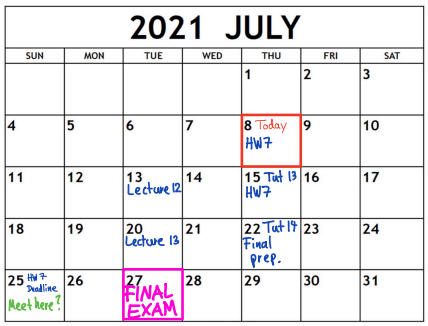
Mathematics Tutorial 2b & Linear Algebra II, Spring semester 2021

Tutorial 12, 8th July 2021, 14:45 - 15:30

Today:
- Continuous dynamical systems
- Homework 7
- Final exam 2020



Tutorial Exercise

Solve the dynamical system $x'(t) = Ax(t)$,

where $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and $x(0) = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

Bonus: What if $x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$?

Let $x : \mathbb{R} \to \mathbb{R}^n$ be a function written as

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

where the entries $x_1, ..., x_n$ are differentiable functions in $C^{(1)}(\mathbb{R}, \mathbb{R})$. By $x'(t) = \frac{d}{dt}x(t)$ we denote

$$x'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}.$$

For a matrix $A \in \mathbb{R}^{n \times n}$ the equation

$$x'(t) = Ax(t)$$

is called a **continuous (linear) dynamical system**.

If A is diagonalizable

We can always solve the $n=1$ case:

Proposition 6.1. *Let $a \in \mathbb{R}$. The only solutions to*

$$x'(t) = a\,x(t)$$

in $C^{(1)}(\mathbb{R}, \mathbb{R})$ are given by $x(t) = c\,e^{at}$ for $c \in \mathbb{R}$.

Strategy to solve $x'(t) = Ax(t)$ $\quad A \in \mathbb{R}^{n \times n}$, $x: \mathbb{R} \to \mathbb{R}^n$

① Diagonalize A

1.1 Find basis $b_1, \dots, b_n$ of eigenvectors of A

1.2 Write $S = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}$

1.3 Then $S^{-1}AS = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

λ_j : eigenvalue of b_j

② Consider the dynamical system $u'(t) = Du(t)$

2.1 Write $u(t) = S^{-1}x(t)$, i.e $x(t) = Su(t)$

2.2 Then $u'(t) = S^{-1}x'(t) = S^{-1}Ax(t) = \underbrace{S^{-1}AS}_{D}\, u(t)$

2.3 The solution of

$$u'(t) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} u(t)$$

is $u(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$ for any $c_1, \dots c_n \in \mathbb{R}$

(3) Get the solution for original system

3.1 Solution is given by

$$X(t) = Su(t) = S\begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

3.2 If you want a solution with $X(0) = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ then you need to find the correct $c_1, \ldots, c_n$ by solving

$$X(0) = S\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

↑ linear system

Homework 7: Dynamical systems & Linear differential equations

Deadline: 25th July (23:55 JST), 2021

Exercise 1. (8 Points) Solve the dynamical system $x'(t) = Ax(t)$, where

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 3 \\ 3 & 2 & 2 \end{pmatrix} \qquad \text{and} \qquad x(0) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} .$$

Exercise 2. (4 Points) Let $F : V \to W$ be a linear map between two vector spaces V and W. Assume that $F(v) = w$ for a fixed $v \in V$ and $w \in W$. Show that the following two statements are equivalent:

i) $F(x) = w$.

ii) $x = v + u$ for some $u \in \ker(F)$.

Exercise 3. (8 Points) Find all solutions to the following differential equation

$$f^{(3)} - f'' - 2f' = 4\,,$$

such that $f(0) = f'(0) = 0$.

(Hint: First, try to find all solutions to the differential equation. For this, find one particular solution (try polynomials) and then consider the homogeneous equation. Then find those solutions out of these such that $f(0) = f'(0) = 0$ holds.)

Tutorial Exercise

(Bonus: Do the $x(0)=\begin{pmatrix}2\\-1\end{pmatrix}$ case)

Solve the dynamical system

$$x'(t) = Ax(t),$$

where $A=\begin{pmatrix}1&2\\2&4\end{pmatrix}$ and $x(0)=\begin{pmatrix}-3\\-1\end{pmatrix}$

① Diagonalize A

$$f_A(\lambda) = \det\begin{pmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{pmatrix} = (1-\lambda)(4-\lambda)-4 = \lambda^2-5\lambda+\overbrace{4-4}^{0}$$
$$= \lambda(\lambda-5)$$

eigenvalues: $\lambda_1=0$, $\lambda_2=5$

$$E_0(A) = \ker\begin{pmatrix}1&2\\2&4\end{pmatrix} = \ker\begin{pmatrix}1&2\\0&0\end{pmatrix} = \operatorname{span}\left\{\underbrace{\begin{pmatrix}-2\\1\end{pmatrix}}_{b_1}\right\}$$

$$E_5(A) = \ker\begin{pmatrix}-4&2\\2&-1\end{pmatrix} = \ker\begin{pmatrix}2&-1\\0&0\end{pmatrix} = \operatorname{span}\left\{\underbrace{\begin{pmatrix}1\\2\end{pmatrix}}_{b_2}\right\}.$$

$$\begin{pmatrix}-4&2\\2&-1\end{pmatrix} \sim \begin{pmatrix}2&-1\\-4&2\end{pmatrix} \sim \begin{pmatrix}2&-1\\0&0\end{pmatrix}$$

$$S = \begin{pmatrix}b_1 & b_2\\ 1 & 1\end{pmatrix} = \begin{pmatrix}-2&1\\1&2\end{pmatrix}$$

$$S^{-1}AS = \begin{pmatrix}0&0\\0&5\end{pmatrix}$$

② Solve $u'(t) = \begin{pmatrix}0&0\\0&5\end{pmatrix}\begin{pmatrix}u_1(t)\\u_2(t)\end{pmatrix}$ ($u(t)$)

$$u_1(t) = c_1 e^{0t} = c_1$$
$$u_2(t) = c_2 e^{5t}$$

$$u(t) = \begin{pmatrix}c_1\\c_2 e^{5t}\end{pmatrix}$$

③ $x(t) = Su(t) = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 e^{5t} \end{pmatrix}$

$= \begin{pmatrix} -2c_1 + c_2 e^{5t} \\ c_1 + 2c_2 e^{5t} \end{pmatrix}$

Want $x(0) = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$: Solve $S \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$\left(\begin{array}{cc|c} -2 & 1 & -3 \\ 1 & 2 & -1 \end{array}\right) \sim \textcircled{2} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ -2 & 1 & -3 \end{array}\right) \sim \frac{1}{5} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 5 & -5 \end{array}\right)$

$\sim \begin{matrix} \\ \textcircled{-2} \end{matrix} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right)$

$\Rightarrow c_1 = 1$

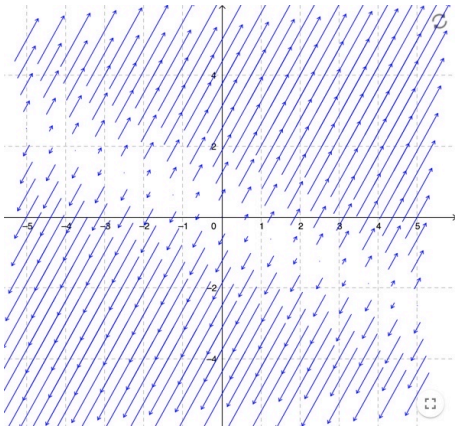
$c_2 = -1$

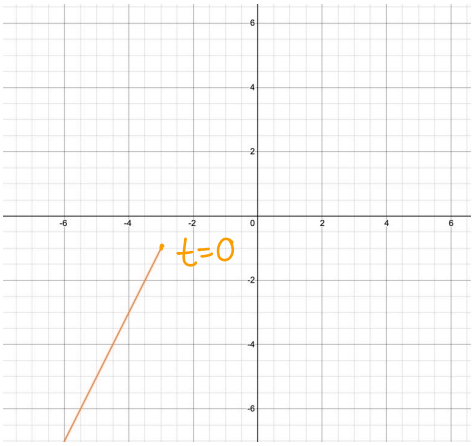
Final solution

$x(t) = \begin{pmatrix} -2 - e^{5t} \\ 1 - 2e^{5t} \end{pmatrix}$

Check:

$Ax(t) = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = x'(t)$





Linear Algebra II & Mathematics Tutorial 2b Spring 2020
Nagoya University, G30 Program Instructor: Henrik Bachmann

Solution Can be found on the homepage

Final exam

Good luck Viel Erfolg 好好儿考啊 시험 잘봐 Semoga sukses амжилт хүсье
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Exercise 1. (14 Points) We define the following matrix

$$A = \begin{pmatrix} 0 & 0 & 2 \\ -3 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}.$$

i) Compute the determinant of A.

ii) Find all eigenvalues of A. For each eigenvalue λ determine a basis of the eigenspace $E_\lambda(A)$.

iii) Is A diagonalizable and/or invertible? Justify your answers.

Exercise 2. (10 Points) Decide if the following statements are true or false. Justify your answer by giving a short explanation.

1) The set $U = \{f \in C^\infty(\mathbb{R},\mathbb{R}) \mid f'' = 2f\}$ is a subspace of $C^\infty(\mathbb{R},\mathbb{R})$.

2) The set $U = \{A \in \mathbb{R}^{2\times 2} \mid \det(A) \neq 0\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

3) If (b_1, b_2) is a basis of a vector space V, then $(b_1 + b_2, -b_2 - b_1)$ is also a basis of V.

4) If $A \in \mathbb{R}^{n\times n}$ is not invertible then A has at least one eigenvalue.

5) There exists a linear map $F : \mathbb{R}^3 \to \mathbb{R}^3$ which satisfies $F \circ F = F$ and which has eigenvalue -1.

Exercise 3. (8 Points) Give an example of

$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $f_B(\lambda) = \lambda^2 + 1$

i) a matrix $A \in \mathbb{R}^{3\times 3}$ which is not diagonalizable and not invertible.

ii) a linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ with $\det(F) = 2$.

iii) a symmetric and orthogonal matrix $B \in \mathbb{R}^{2\times 2}$ which is not the identity matrix.

Justify your examples.

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} \square & 0 \\ 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Exercise 4. (10 Points) Let $x_n \in \mathbb{R}^2$ be for $n \geq 0$ be defined by

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad x_{n+1} = Mx_n\,, \quad \text{where} \quad M = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

Determine an explicit formula for x_n.

$x_n = M^n x_0$

Exercise 5. (8 Points) Find all solutions to the following differential equation

$$f''' - 2f'' + f' = 2020\,.$$

After finishing this exam please send your solution as one pdf file to
henrik.bachmann@math.nagoya-u.ac.jp.