Mathematics Tutorial 2b & Linear Algebra II, Spring semester 2021

Tutorial 10, 24th June 2021, 14:45 - 15:30

Today:

- Orthogonal maps/matrices and Orthonormal bases
- Homework 5
- Homework 6

Definition 5.11. *An* **orthogonal map** *is a linear map* $F : \mathbb{R}^n \to \mathbb{R}^n$, *such that*

$$\|F(x)\| = \|x\|, \qquad \forall x \in \mathbb{R}^n,$$

i.e. the map F *does not change the norm of a vector. We call a matrix* $A \in \mathbb{R}^{n\times n}$ *orthogonal if* $\|Ax\| = \|x\|$ *for all* $x \in \mathbb{R}^n$.

Theorem 5.14. *Let* $F : \mathbb{R}^n \to \mathbb{R}^n$ *a linear map and* $A = [F]_B$ *the matrix of* F *for* $B = (e_1, \dots, e_n)$. *The following statements are equivalent.*

i) F *is orthogonal.*

ii) A *is orthogonal.*

iii) For all $x, y \in \mathbb{R}^n$ *we have* $F(x) \bullet F(y) = x \bullet y$.

iv) A *is invertible and* $A^{-1} = A^T$.

v) $(F(e_1), \dots, F(e_n))$ *(the columns of* A*) is an orthonormal basis of* $\mathbb{R}^n$.

vi) If $(b_1, \dots, b_n)$ *is an orthonormal basis of* $\mathbb{R}^n$ *then* $(F(b_1), \dots, F(b_n))$ *is also an orthonormal basis.*

Tutorial Exercise

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Find an orthogonal matrix S, such that

$$S^T A S = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

for some $d_1, d_2 \in \mathbb{R}$.

LA I (Lecture 12)

☺ The norm of $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ is $\|x\| = \sqrt{x_1^2 + \dots + x_n^2}$

☺ For $x, y \in \mathbb{R}^n$ we define the dot product by

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$x \bullet y = x_1 y_1 + \dots + x_n y_n = (x_1 \cdots x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x^T y.$$

$\|x\| = \sqrt{x \bullet x}$ $\qquad \|\lambda x\| = |\lambda| \, \|x\|$

☺ Two vectors $x, y \in \mathbb{R}^n$ are called orthogonal if

$$x \bullet y = 0.$$

$B = (e_1, \dots, e_n)$ is an ONB of $\mathbb{R}^n$.

☺ A basis $B = (b_1, \dots, b_m)$ of a subspace $U \subset \mathbb{R}^n$ is an orthonormal basis (ONB) if

$$b_i \bullet b_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

e.g. $B = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ is an ONB of $\mathbb{R}^2$

Notice that $v = 0$ always satisfies (5.1) for any $\lambda \in \mathbb{R}$, since F is a linear map. This is one of many reasons why $v = 0$ is not called an eigenvector of F.

In the following, we always assume that V is a finitely generated vector space.

Definition 5.2. *Let $F : V \to V$ be a linear map let $\mathrm{id}_V : V \to V$ be the identity map on V.*

i) The polynomial $f_F(\lambda) = \det(F - \lambda \,\mathrm{id}_V)$ is called the **characteristic polynomial** *of F.*

ii) Let $\lambda \in \mathbb{R}$ be an eigenvalue of F. Then the space

$$\begin{aligned} E_\lambda(F) &= \ker(F - \lambda \,\mathrm{id}_V) \\ &= \{v \in V \mid F(v) = \lambda v\} \end{aligned}$$

is called the **eigenspace** *of F with respect to the eigenvalue λ.*

The eigenspace $E_\lambda(F)$ contains therefore all eigenvectors of F with eigenvalue λ and the zero vector.

Lecture 8

Definition 5.3. *i) Let $\dim V = n$. A linear map $F : V \to V$ is called* **diagonalizable** *if there exist a basis B of V, such that*

$$[F]_B = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

for some $d_1, \ldots, d_n \in \mathbb{R}$.

ii) A matrix $A \in \mathbb{R}^{n \times n}$ is called **diagonalizable** *if there exists an invertible matrix $S \in \mathbb{R}^{n \times n}$ with*

$$S^{-1}AS = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

for some $d_1, \ldots, d_n \in \mathbb{R}$.

Lemma 5.4. *Let B be a basis of V and let $F : V \to V$ be a linear map. Then the following two statements are equivalent*

i) The linear map F is diagonalizable.

ii) The matrix $[F]_B$ is diagonalizable.

Lemma 5.5. *Let $F : V \to V$ be a linear map and $B = (b_1, \ldots, b_n)$ be a basis of V, such that all b_i are eigenvectors of F, i.e. $F(b_i) = d_i b_i$ for some $d_i \in \mathbb{R}$ and $i = 1, \ldots, n$. Then F is diagonalizable and*

$$[F]_B = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

Conversely, if F is diagonalizable then there exists a basis of eigenvectors.

Theorem 5.6. *Let $v_1, \dots, v_m \in V$ be eigenvectors of a linear map $F : V \to V$ with different eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Then $v_1, \dots, v_m$ are linearly independent.*

Corollary 5.7. *Let $F : V \to V$ be a linear map with eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and $\dim V = n$.*

i) If F has n distinct eigenvalues, i.e. $m = n$, then F is diagonalizable.

ii) If $B_1, \dots, B_m$ are bases of $E_{\lambda_1}(F), \dots, E_{\lambda_m}(F)$, then $B_1 \cup \dots \cup B_m$ are linearly independent.

iii) The map F is diagonalizable if and only if

$$\sum_{j=1}^{m} \dim E_{\lambda_j}(F) = n\,.$$

Definition 5.8. *Let $F : V \to V$ be a linear map and let $\lambda \in \mathbb{R}$ be an eigenvalue of F.*

i) The **algebraic multiplicity** *of λ, denoted by $\operatorname{algmu}_F(\lambda)$, is the multiplicity of λ in the characteristic polynomial f_F.*

ii) The **geometric multiplicity** *of λ is given by $\operatorname{geomu}_F(\lambda) = \dim E_\lambda(F)$.*

Theorem 5.9. *Let $F : V \to V$ be a linear map and $\lambda \in \mathbb{R}$ be an eigenvalue of F. Then*

$$\operatorname{geomu}_F(\lambda) \leq \operatorname{algmu}_F(\lambda)\,.$$

Corollary 5.10. *If F is diagonalizable then $\operatorname{geomu}_F(\lambda) = \operatorname{algmu}_F(\lambda)$ for all eigenvalues λ of F.*

Lecture 9

5.1 The spectral theorem

In this section we will just consider the vector space $V = \mathbb{R}^n$. Recall that the **norm** of a vector $x \in \mathbb{R}^n$ is defined by

$$\|x\| = \sqrt{x_1^2 + \dots + x_n^2}\,, \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n\,.$$

Definition 5.11. *An* **orthogonal map** *is a linear map $F : \mathbb{R}^n \to \mathbb{R}^n$, such that*

$$\|F(x)\| = \|x\|\,, \qquad \forall x \in \mathbb{R}^n\,,$$

i.e. the map F does not change the norm of a vector. We call a matrix $A \in \mathbb{R}^{n\times n}$ orthogonal if $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.

Recall that the **dot product** $\bullet$ for two vectors $x, y \in \mathbb{R}^n$ is defined by

$$x \bullet y = x^T y = x_1 y_1 + \dots + x_n y_n \in \mathbb{R}\,.$$

With this the norm of a vector can also be written as $\|x\| = \sqrt{x \bullet x}$.

Lemma 5.12. *For all $x, y \in \mathbb{R}^n$ we have*

$$x \bullet y = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right) .$$

Proposition 5.13. *A linear map $F : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if*

$$F(x) \bullet F(y) = x \bullet y$$

for all $x, y \in \mathbb{R}^n$.

Recall: We say that x and y are **orthogonal** if $x \bullet y = 0$. A basis $B = (b_1, \ldots, b_n)$ of $\mathbb{R}^n$ is called an **orthonormal basis** if b_i and b_j for $i \neq j$ are orthogonal and $\|b_i\| = 1$ for all i, i.e.

$$b_i \bullet b_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} .$$

Theorem 5.14. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ a linear map and $A = [F]_B$ the matrix of F for $B = (e_1, \ldots, e_n)$. The following statements are equivalent.*

i) F is orthogonal.

ii) A is orthogonal.

iii) For all $x, y \in \mathbb{R}^n$ we have $F(x) \bullet F(y) = x \bullet y$.

iv) A is invertible and $A^{-1} = A^T$.

v) $(F(e_1), \ldots, F(e_n))$ (the columns of A) is an orthonormal basis of $\mathbb{R}^n$.

vi) If $(b_1, \ldots, b_n)$ is an orthonormal basis of $\mathbb{R}^n$ then $(F(b_1), \ldots, F(b_n))$ is also an orthonormal basis.

Corollary 5.15. *i) $A \in \mathbb{R}^{n\times n}$ is orthogonal if and only if A^T is orthogonal.*

ii) If $A, B \in \mathbb{R}^{n\times n}$ are orthogonal then AB is orthogonal.

iii) If B_1 and B_2 are two orthonormal bases, then the change of basis matrix $S_{B_1}^{B_2}$ is orthogonal.

Definition 5.16. *i) An* **eigenbasis** *of a linear map $F : \mathbb{R}^n \to \mathbb{R}^n$ is a basis consisting of eigenvectors of F.*

ii) Let $U \subset \mathbb{R}^n$ be a subspace. A linear map $F : U \to U$ is called **symmetric** *if we have for all $x, y \in U$*

$$x \bullet F(y) = F(x) \bullet y .$$

$\Longleftrightarrow$ $[F]_B^T = [F]_B$ B ONB of U

Theorem 5.17. *(Spectral theorem) Let $U \subset \mathbb{R}^n$ be a subspace and $F : U \to U$ a linear map. Then F is symmetric if and only if there exists an orthonormal eigenbasis of F.*

Corollary 5.18. *A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if and only if there exists an orthogonal matrix $S \in \mathbb{R}^{n\times n}$, such that*

$$S^T A S = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix},$$

for some $d_1, \dots, d_n \in \mathbb{R}$.

$S = \begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix}$

$\mathcal{B} = (b_1 \dots b_n)$ is an orthonormal eigenbasis of A.

Lemma 5.19. *Every symmetric linear map $F : U \to U$ has an eigenvalue.*

Tutorial Exercise

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Find an orthogonal matrix S, such that $S^T A S = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$ for some $d_1, d_2 \in \mathbb{R}$.

1) Find eigenvalues:

$f_A(\lambda) = \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix}$ $(\lambda+a)(\lambda+b)$ $\lambda^2 + (a+b)\lambda + ab$

$= (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$

$= (\lambda-3)(\lambda-1).$

$\lambda_1 = 3, \ \lambda_2 = 1.$

2) $E_3(A) = \ker \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \ker \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$

$E_1(A) = \ker \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$ eigenbasis but not ONB

$\left\| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \sqrt{2} \neq 1$

$\tilde{\mathcal{B}} = \left(\underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{b_1}, \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{b_2} \right)$ is an ONB

Thm. 5.14

$S = \begin{pmatrix} | & | \\ b_1 & b_2 \\ | & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Longrightarrow S$ is orthogonal

$S^{-1} = S^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$S^T A S = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$

Homework 5: Eigenvalues & Eigenvectors II

Deadline: 27th June (23:55 JST), 2021

Exercise 1. (6 Points) We define the sequence $(a_n)_{n\geq 0}$ recursivley by $a_0 = 0$, $a_1 = 1$ and for $n \geq 0$ by

$$a_{n+2} = 2a_n + a_{n+1}\,.$$

Determine an explicit formula for a_n by using diagonalization.
(Hint: Compare this to the Fibonacci number example in Homework 2 Exercise 1)

Exercise 2. (4+4 = 8 Points)

i) Let $U \subset \mathbb{R}^n$ be a subspace and let $P_U : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection to U. Show that P_U is diagonalizable. What are the eigenvalues of P_U?
(See Linear Algebra I Section 12 & 13 for the definition of P_U)

ii) Let $\mathrm{rot}_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by an angle $\alpha \in [0, 2\pi]$. For which α is rot_α diagonalizable?

Exercise 3. (4+4 = 8 Points) Let $F : V \to V$ be a linear map with eigenvalues $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and $\dim V = n$. Show the following statements by using Theorem 5.6:

i) If $B_1, \ldots, B_m$ are bases of $E_{\lambda_1}(F), \ldots, E_{\lambda_m}(F)$, then $B_1 \cup \cdots \cup B_m$ are linearly independent.
(Here we mean by $B_1 \cup \cdots \cup B_m$ the collection of all vectors in the bases $B_1, \ldots, B_m$.)

ii) The map F is diagonalizable if and only if

$$\sum_{j=1}^{m} \dim E_{\lambda_j}(F) = n\,.$$

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Linear Algebra II & Mathematics Tutorial 2b Spring 2021
Nagoya University, G30 Program Instructor: Henrik Bachmann

Homework 6: Eigenvalues & Eigenvectors III

Deadline: 4th July (23:55 JST), 2021

Exercise 1. (2+4+2 = 8 Points) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

i) Calculate all eigenvalues of A.

ii) Determine orthonormal bases for all eigenspaces of A.
(Recall the Gram-Schmidt algorithm from Linear Algebra I for this)

iii) Find an orthogonal matrix $S \in \mathbb{R}^{3\times 3}$, such that $S^T AS$ is a diagonal matrix.

Exercise 2. (3+3 = 6 Points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with $A = [F]_B$, where $B = (e_1, \dots, e_n)$. Show the following two implications:

i) If $(F(e_1), \dots, F(e_N))$ is an orthonormal basis then A is invertible and $A^{-1} = A^T$.

ii) If $A^{-1} = A^T$ then $F(x) \bullet F(y) = x \bullet y$ for all $x, y \in \mathbb{R}^n$.

columns of A.

$\rightsquigarrow$ calculate $AA^T = \dots = I_n$

$\| \quad (Ax)\bullet(Ay) = \dots = x \bullet y$

Exercise 3. (8 Points) Show that a matrix $A \in \mathbb{R}^{2\times 2}$ is symmetric, i.e. $A = A^T$, if and only if there exists an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

(This is the $n = 2$ case of the spectral theorem (Theorem 5.17), which you are not allowed to use to solve this Exercise. You are also not allowed to use Lemma 5.19 and need to prove it in this special case.)

"$\Rightarrow$" Assume A sym., $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$

..... find orthonormal basis of $\mathbb{R}^2$ made out of eigenvectors of A.

"$\Leftarrow$" Have ONB $B = (b_1, b_2)$ where b_1, b_2 are eigenvectors of A

..... A is symmetric.

Version: June 24, 2021

The Gram-Schmidt algorithm / process

This is part of the Lecture notes 12 in LAI

For a subspace V with basis $\mathcal{B}$ we want to create an ONB $\mathcal{F}$ of V.

Youtube: LA1, Lecture 12 50:00~...

Example 25 Consider $b_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $b_2 = \begin{pmatrix}1\\0\\2\end{pmatrix}$, $b_3 = \begin{pmatrix}-1\\2\\1\end{pmatrix}$.

$\mathcal{B} = (b_1, b_2, b_3)$ is a basis of $\mathbb{R}^3$.

Want to construct ONB $\mathcal{F} = (f_1, f_2, f_3)$. $\left(f_i \cdot f_j = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}\right)$

Step 1: $f_1 = \hat{b}_1 = \frac{1}{\|b_1\|} b_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$.

Step 2.1: Make b_2 orthogonal to f_1

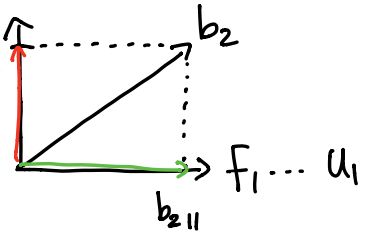
Set $U_1 = \operatorname{span}\{f_1\} = \operatorname{span}\{b_1\}$

Lemma 12.5 iv) $\Rightarrow$ $b_2 = \underset{\in U_1}{b_{2\parallel}} + \underset{\in U_1^\perp}{b_{2\perp}} = (b_2 \cdot f_1) f_1 + b_{2\perp}$

$\begin{pmatrix}1\\0\\2\end{pmatrix} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{3}} \cdot 3$

$w_2 = b_{2\perp} = b_2 - (b_2 \cdot f_1) f_1$ $\quad w_2 = b_{2\perp}$

$= \begin{pmatrix}1\\0\\2\end{pmatrix} - \frac{3}{\sqrt{3}} f_1 = \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\-1\\1\end{pmatrix}$.



Step 2.2: Set $f_2 = \hat{w}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$.

Set $U_2 = \text{span}\{f_1, f_2\} = \text{span}\{b_1, f_2\}$

$= \text{span}\{b_1, b_2\}$.

(f_1, f_2) are ONB of U_2.

Step 3.1 $\quad b_3 = \underset{\in U_2}{b_{3\parallel}} + \underset{\in U_2^\perp}{b_{3\perp}}$

$\left(\begin{smallmatrix}-1\\2\\1\end{smallmatrix}\right) \cdot \frac{1}{\sqrt{3}}\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right) = \frac{1}{\sqrt{3}} \cdot 2 \qquad \left(\begin{smallmatrix}-1\\2\\1\end{smallmatrix}\right) \cdot \frac{1}{\sqrt{2}}\left(\begin{smallmatrix}0\\-1\\1\end{smallmatrix}\right) = \frac{1}{\sqrt{2}} \cdot (-1)$

Set $w_3 = b_{3\perp} = b_3 - (b_3 \cdot f_1) f_1 - (b_3 \cdot f_2) f_2$

$$= \begin{pmatrix}-1\\2\\1\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\-1\\1\end{pmatrix} = \frac{5}{6}\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

Step 3.2 $\quad$ Set $f_3 = \hat{w}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix}-2\\1\\1\end{pmatrix}$.

$U_3 = \text{span}\{f_1, f_2, f_3\} = \text{span}\{b_1, b_2, f_3\} = \text{span}\{b_1, b_2, b_3\}$.

$F = (f_1, f_2, f_3)$ are ONB of $U_3 = \mathbb{R}^3$.