

Review for the final exam

The following exercises are for the preparation of the final exam. The final exam will contain less exercises!

Exercise 1. We define the following matrix

$$A = \begin{pmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

- i) Compute the determinant of A .
- ii) Find all eigenvalues of A . For each eigenvalue λ determine a basis of the eigenspace $E_\lambda(A)$.
- iii) Is A diagonalizable and/or invertible and/or orthogonal? Justify your answers.

Exercise 2. Decide if the following statements are true or false. Justify your answer by giving a short explanation (e.g. give a counterexample in the case when it is false). V is always a finitely generated vector space and $A, B \in \mathbb{R}^{n \times n}$ are matrices.

- 1) The set $U = \{p \in \mathcal{P}_2 \mid p(1) = p(2)\}$ is a subspace of \mathcal{P}_2 .
- 2) The set $U = \{A \in \mathbb{R}^{2 \times 2} \mid A \text{ has eigenvalue } 1\}$ is a subspace of $\mathbb{R}^{2 \times 2}$.
- 3) Every 3×3 matrix has at least one eigenvalue.
- 4) For any subspace $U \subset V$, there exist a linear map $F : V \rightarrow V$ with $\ker(F) = U$.
- 5) There exists a linear map $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with $\dim(\ker(F)) = \dim(\text{im}(F)) = 2$.
- 6) If (b_1, b_2) and (c_1, c_2) are two bases of a vector space V , then $(b_1 + c_1, b_2 - c_2)$ is also a basis of V .
- 7) If $\det(A) = 0$, then A has at least one eigenvalue.
- 8) If $\det(A) = 0$ and $\det(B) = 0$ then $\det(A + B) = 0$.
- 9) There exist symmetric matrices without eigenvalues.
- 10) Every orthogonal matrix is invertible.
- 11) Let $F : V \rightarrow V$ be a linear map. If $F \circ F$ has at least one eigenvalue, then F also has at least one eigenvalue.

Exercise 3. Give an example of

- i) a linear map $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $\dim(\ker(F)) = \dim(\text{im}(F)) = 1$.
- ii) a matrix A which is not symmetric but diagonalizable.
- iii) a linear map $F : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ without eigenvalues.
- iv) an orthogonal matrix $A \in \mathbb{R}^{2 \times 2}$ without 0 as an entry.

Exercise 4. Let $x_n \in \mathbb{R}^2$ be for $n \geq 0$ be defined by

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{n+1} = Mx_n, \quad \text{where} \quad M = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

Determine an explicit formula for x_n .

Exercise 5. Let $F : \mathcal{P}_2 \rightarrow \mathcal{P}_2$ be the linear map given by

$$F(p)(t) = tp'(t) + 2p(t).$$

Calculate the determinant of F .

Exercise 6. Let V be a finitely generated vector space and $P : V \rightarrow V$ be a linear map with $P \circ P = P$. Assume that λ is an eigenvalue of P . What are the possible values of λ ?

Exercise 7. Find all solutions to the following differential equation

$$f''' - 2f'' = 4,$$

such that $f(0) = f'(0) = f''(0) = 0$.