Solution to the Review for the final exam

The following exercises are for the preparation of the final exam. The final exam will contain less exercises!

Exercise 1. We define the following matrix

$$A = \begin{pmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

i) Compute the determinant of A.

By Laplace Expansion over the 1^{st} column we get

$$\begin{aligned} \det(A) &= (1)\det\begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix} - (0)\det\begin{pmatrix} 5 & 4 \\ 4 & 2 \end{pmatrix} + (0)\det\begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} \\ &= (2\cdot 2) - 0 - 0 + 0 = \underline{\underline{4}}. \end{aligned}$$

ii) Find all eigenvalues of A. For each eigenvalue λ determine a basis of the eigenspace $E_\lambda(A)$.

We find the characteristic polynomial of A by considering the determinant of

$$A - \lambda I_3 = \begin{pmatrix} 1-\lambda & 5 & 4 \\ 0 & 2-\lambda & 0 \\ 0 & 3 & 2-\lambda \end{pmatrix}$$

By Laplace Expansion over the 1^{st} column we get

$$\begin{aligned} \det(A-\lambda I_3) &= (1-\lambda)\det\begin{pmatrix} 2-\lambda & 0 \\ 3 & 2-\lambda \end{pmatrix} - (0)\det\begin{pmatrix} 5 & 4 \\ 3 & 2-\lambda \end{pmatrix} + (0)\det\begin{pmatrix} 5 & 4 \\ 2-\lambda & 0 \end{pmatrix} \\ &= (1-\lambda)(2-\lambda)(2-\lambda) \end{aligned}$$

Which means that the eigenvalues of A are

$$\lambda = \underline{\underline{1}} \quad \text{and} \quad \lambda = \underline{\underline{2}}.$$

For $\lambda = 1$, let $v \in \ker(A - I_3) = E_1(A)$. Then v satisfies

$$\begin{pmatrix} 0 & 5 & 4 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 5v_2 + 4v_3 \\ v_2 \\ 3v_2 + v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the three linear equations give the result that

$$\ker(A - I_3) = E_1(A) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} \iff \underline{\underline{B_1 = \left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\} \text{ is a basis of } E_1(A).}}$$

For $\lambda = 2$, let $v \in \ker(A - 2I_3) = E_2(A)$. Then v satisfies

$$\begin{pmatrix} -1 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} -v_1 + 5v_2 + 4v_3 \\ 0 \\ 3v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the three linear equations give the result that

$$\ker(A - 2I_3) = E_2(A) = \operatorname{span}\left\{\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}\right\} \iff \underline{\underline{B_2 = \left\{\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}\right\} \text{ is a basis of } E_2(A).}}$$

iii) Is A diagonalizable and/or invertible and/or orthogonal ? Justify your answers.

A is invertible since $\det(A) = 4 \neq 0$.

A is not diagonalizable since we cannot find an eigenbasis of A (as $\text{algmu}_A(2) = 2 > 1 = \text{geomu}_A(2)$)

A is not orthogonal since $\begin{pmatrix}1\\0\\0\end{pmatrix} \bullet \begin{pmatrix}4\\0\\2\end{pmatrix} \neq 0$.

Exercise 2. Decide if the following statements are true or false. Justify your answer by giving a short explanation (e.g. give a counterexample in the case when it is false). V is always a finitely generated vector space and $A, B \in \mathbb{R}^{n\times n}$ are matrices.

1) The set $U = \{p \in \mathcal{P}_2 \mid p(1) = p(2)\}$ is a subspace of $\mathcal{P}_2$.

   True: Observe that the set U satisfies (three conditions of a subspace):

   i) Let the neutral element of $\mathcal{P}_2$ be $n(x) = 0$. Then $n(1) = n(2) = 0$, so $n \in U$.

   ii) Let p and q be elements in U. Then we observe that

   $$(p+q)(1) \;=\; p(1)+q(1) \;=\; p(2)+q(2) \;=\; (p+q)(2) \iff (p+q) \in U$$

   iii) Let $p \in U$ and $\lambda \in \mathbb{R}$. Then we observe that

   $$(\lambda p)(1) \;=\; \lambda \cdot p(1) \;=\; \lambda \cdot p(2) \;=\; (\lambda p)(2) \iff (\lambda p) \in U$$

   From which we conclude that U is a subspace of $\mathcal{P}_2$.

2) The set $U = \{A \in \mathbb{R}^{2\times 2} \mid A \text{ has eigenvalue } 1\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

   False: Observe that $I_2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \in U$, but $2I_2 = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix} \notin U$ as $2I_2$ only has eigenvalue 2.

3) Every 3×3 matrix has at least one eigenvalue.

   True: The Characteristic Polynomial of a 3×3 matrix is a 3^{rd} degree polynomial. Every odd-degree polynomial always has at least one real zero (and therefore has at least one eigenvalue).

4) For any subspace $U \subset V$, there exist a linear map $F : V \to V$ with $\ker(F) = U$.

   True: As V is finitely generated, then U is also finitely generated. Let $B_U = (u_1, \ldots, u_m)$ be a basis of U. Then we can extend this to a basis of V, $B_V = (u_1, \ldots, u_m, v_1, \ldots, v_l)$. Now we define the linear map F by $F(u_j) = 0$ and $F(v_i) = v_i$ for $j \in [1, m]$ and $i \in [1, l]$. Then $\ker(F) = U$.

5) There exists a linear map $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $\dim(\ker(F)) = \dim(\text{im}(F)) = 2$.

   False: By **Theorem 2.3.** we have that for a linear map $F : V \to V$

   $$\dim(V) = \dim(\ker(F)) + \dim(\text{im}(F))$$

   And therefore here, $\dim(\mathbb{R}^3) = 3 = \dim(\ker(F)) + \dim(\text{im}(F)) \neq 2 + 2$, disproving the statement.

6) If (b_1, b_2) and (c_1, c_2) are two bases of a vector space V, then $(b_1 + c_1, b_2 - c_2)$ is also a basis of V.

   False: Let $(b_1, b_2) = (c_1, c_2)$. Then $(b_1 + c_1, b_2 - c_2) = (2b_1, n_V)$ which is not a basis of V as n_V is linearly dependent with any other vector.

7) If $\det(A) = 0$, then A has at least one eigenvalue.

   True: $\det(A) = 0$ implies that $\ker(A) \neq \{0\}$. This means that $\exists\, v \in \mathbb{R}^n$ such that $Av = 0$, implying that 0 is an eigenvalue of A.

8) If $\det(A) > 0$, then A has at least one eigenvalue.

False: Consider the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\det(A) = 1$. The characteristic polynomial of A is $\lambda^2 + 1$. As such, A does not have any (real-valued) eigenvalue (as there exists no solutions to $\lambda^2 + 1 = 0$).

9) If $\det(A) = 0$ and $\det(B) = 0$ then $\det(A + B) = 0$.

False: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$. Here, observe that $\det(A) = \det(B) = 0$ but $\det(A + B) = \det(I_2) = 1 \neq 0$.

10) There exist symmetric matrices without eigenvalues.

False: By the **Spectral Theorem**, all symmetric matrices are diagonalizable (and as such exists an eigenbasis of said symmetric matrices, implying the existence of at least one eigenvalue).

11) Every orthogonal matrix is invertible.

True: For an orthogonal matrix A, its inverse is given by A^T.

12) Let $F : V \to V$ be a linear map. If $F \circ F$ has at least one eigenvalue, then F also has at least one eigenvalue.

False: Consider the linear map

$$\begin{aligned} F : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \end{aligned}$$

And then we have that $F \circ F(x)$ is given by

$$F \circ F(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x$$

Observe that $F \circ F$ has eigenvalue -1.
However, F has no eigenvalues as the characteristic polynomial of F is equal to

$$p_F(\lambda) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1$$

And there exists no real-valued λ such that $\lambda^2 + 1 = 0$.

Exercise 3. Give an example of

i) a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ with $\dim(\ker(F)) = \dim(\operatorname{im}(F)) = 1$.

We consider the linear map

$$\begin{aligned} F : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\longmapsto \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \end{aligned}$$

This implies that $\ker(F) = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $\operatorname{im}(F) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.
Therefore $\dim \ker(F) = \dim \operatorname{im}(F) = 1$.

ii) a matrix A which is not symmetric but diagonalizable.

Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. We have that the characteristic polynomial of A is

$$p_A(\lambda) = \det(A - \lambda I_2) = \det \begin{pmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda(\lambda - 1)$$

From which we conclude that the eigenvalues of A are $\lambda = 0$ and $\lambda = 1$
By **Corollary 5.7** we have that since A has 2 eigenvalues (and $\dim \mathbb{R}^2 = 2$) then $\underline{\underline{A \text{ is diagonalizable}}}$.

iii) a linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ without eigenvalues.
Consider the linear map

$$\begin{aligned} F : \mathcal{P}_1 &\longrightarrow \mathcal{P}_1 \\ ax + b &\longmapsto -bx + a \end{aligned}$$

Consider $B = (x, 1)$ which is one possible basis of $\mathcal{P}_2$.
Then we have that

$$[F]_B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

And then we observe that the characteristic polynomial of F is $P_F(\lambda) = \lambda^2 + 1$
From which we conclude that $\underline{\underline{F \text{ has no eigenvalues.}}}$

iv) an orthogonal matrix $A \in \mathbb{R}^{2\times 2}$ without 0 as an entry.
$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is orthogonal since

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \bullet \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}((-1) + 1) = 0$$

and we have that

$$\left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\| = \left\| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\| = 1 \, .$$

Exercise 4. Let $x_n \in \mathbb{R}^2$ be for $n \geq 0$ be defined by

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad x_{n+1} = M x_n \, , \qquad \text{where} \qquad M = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} .$$

Determine an explicit formula for x_n.
We try to diagonalize M. First we find the characteristic polynomial of M which is given by:

$$p_M(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 \\ 3 & 3-\lambda \end{pmatrix} = (\lambda - 1)(\lambda - 3) - 3 = \lambda^2 - 4\lambda = \lambda(\lambda - 4)$$

From which we find that the eigenvalue of M are

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 4 \, .$$

Then we find the eigenvectors (and the eigenbasis) of M:
For $\lambda = 0$, let $v \in E_0(M) = \ker(M)$. Then v satisfies

$$\begin{aligned} \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} v_1 + v_2 \\ 3v_1 + 3v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

From which we get the result that

$$\ker(M) = E_0(M) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

For $\lambda = 4$, let $v \in E_4(M) = \ker(M - 4I_2)$. Then v satisfies

$$\begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -3v_1 + v_2 \\ 3v_1 - v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From which we get the result that

$$\ker(M) = E_0(M) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$$

We then conclude that $B = \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right)$ is an eigenbasis of M.

Thus if we set $S := \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ we have that

$$S^{-1}MS = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$

We can find the inverse of S, S^{-1} by:

$$S^{-1} = \frac{1}{\det(S)} \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3+1} \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

We observe that

$$M = S \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} S^{-1} \iff M^n = S \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}^n S^{-1}$$

As $S^{-1}S = SS^{-1} = I_2$. We observe that $M^n x_0 = x_n$, so we get that

$$\begin{aligned} x_n &= S \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}^n S^{-1} x_0 \\ &= \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 4^n \end{pmatrix} \frac{1}{4} \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 0 & 4^n \\ 0 & 3 \cdot 4^n \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 4^n & -4^n \\ 3 \cdot 4^n & -3 \cdot 4^n \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 4^n \\ 3 \cdot 4^n \end{pmatrix} = \begin{pmatrix} 4^{n-1} \\ 3 \cdot 4^{n-1} \end{pmatrix}. \end{aligned}$$

Remark: This formula is valid for $n \geq 1$, and isn't valid for $n = 0$, as x_0 is defined arbitrarily.

Exercise 5. Let $F : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear map given by

$$F(p)(t) = tp'(t) + 2p(t)\,.$$

Calculate the determinant of F.

If we want to find $\det(F)$ we can define a basis B of $\mathcal{P}_2$ and then we have that $\det[F]_B = \det(F)$. Let $B = (b_1, b_2, b_3) = (t^2, t, 1)$. Then to find $[F]_B$ we find:

$$\begin{aligned} F(b_1) &= t(2t) + 2(t^2) = 4t^2 = 4b_1 \\ F(b_2) &= t(1) + 2(t) = 3t = 3b_2 \\ F(b_3) &= t(0) + 2(1) = 2 = 2b_3 \end{aligned}$$

From which we have the result that

$$[F]_B = \begin{pmatrix} | & | & | \\ [F(b_1)]_B & [F(b_2)]_B & [F(b_3)]_B \\ | & | & | \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \implies \det(F) = \det[F]_B = \underline{\underline{24}}\,.$$

Exercise 6. Let V be a finitely generated vector space and $P : V \to V$ be a linear map with $P \circ P = P$. Assume that λ is an eigenvalue of P. What are the possible values of λ?
Let $v \in V$ $(v \neq n_V)$ be an eigenvector of P with eigenvalue λ. Then,

$$P(v) = \lambda \cdot v$$

And the equality $P \circ P = P$ implies that

$$\begin{aligned} & P \circ P(v) = P(v) \\ \iff & P(\lambda \cdot v) = \lambda \cdot v \\ \iff & \lambda \cdot \lambda \cdot v = \lambda v \\ \iff & (\lambda^2 - \lambda)v = n_V \\ \iff & \lambda(\lambda - 1)v = n_V \end{aligned}$$

Since $v \neq n_V$ we get that $\lambda(\lambda - 1) = 0$. As such, the only possible values of λ are:

$$\lambda = \underline{\underline{0}} \quad \text{and} \quad \lambda = \underline{\underline{1}}\,.$$

Exercise 7. Find all solutions to the following differential equation

$$f''' - 2f'' = 4\,,$$

such that $f(0) = f'(0) = f''(0) = 0$.

We have the differential operator

$$T(f) = f''' - 2f''$$

And want to find the solutions to $T(f) = 4$.
First we find one particular solution. $f_p(t) = -t^2$ is a solution as

$$f_p'''(t) - 2f_p''(t) = 0 - 2(-2) = 4$$

Now we determine the kernel of T by finding the characteristic polynomial of T:

$$\begin{aligned} P_T(x) &= x^3 - 2x^2 \\ &= x^2(x-2) \end{aligned}$$

Therefore the zeroes of the characteristic polynomial are 0 and 2.
Then we know that the kernel of T is given by

$$\ker(T) = \text{span}\{e^{0t}, te^{0t}, e^{2t}\} = \text{span}\{1, t, e^{2t}\}$$

Which means that all solutions to $T(f) = 4$ are given by

$$f(t) = -t^2 + c_1 + c_2 t + c_3 e^{2t} \quad \text{for } c_1, c_2, c_3 \in \mathbb{R}\,.$$

Now we want to find the solutions that satisfy $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 0$:

$$\begin{aligned} f(0) &= 0 \iff 0 = -(0)^2 + c_1 + c_2(0) + c_3(1) = c_1 + c_3 \\ f'(0) &= 0 \iff 0 = (-2)(0) + 0 + c_2 + c_3(2)(1) = c_2 + 2c_3 \\ f''(0) &= 0 \iff 0 = -2 + 0 + 4c_3(1) = -2 + 4c_3 \end{aligned}$$

This corresponds to the matrix equation

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Solving these three linear equations gives us the result that

$$c_3 = \frac{1}{2} \qquad c_2 = -2c_3 = -1 \qquad c_1 = -c_3 = -\frac{1}{2}\,.$$

Which means that

$$f(t) = \underline{\underline{\frac{1}{2}e^{2t} - t^2 - t - \frac{1}{2}}}$$

is a solution to $T(f) = 4$ and satisfies $f(0) = f'(0) = f''(0) = 0$.

終わり