Linear Algebra I - Midterm preparation

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Nagoya University, G30 Program

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Possible midterm type exercises from the homework: HW1 Ex. 1, Ex. 2, HW2 Ex. 2.

1) Decide if the following statements are true or false. Justify your answer by giving a short explanation.

i) The set $U = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f'' = 2f\}$ is a subspace of $C^\infty(\mathbb{R}, \mathbb{R})$.

ii) The set $U = \left\{A \in \mathbb{R}^{2\times 2} \mid \det(A) \neq 0\right\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

iii) If (b_1, b_2) is a basis of a vector space V, then $(b_1 + b_2, b_2 - b_1)$ is also a basis of V.

iv) If (f_1, f_2, f_3) is a basis of $\mathcal{P}_2$, then (f_1', f_2', f_3') is also a basis of $\mathcal{P}_2$.

v) For any real number $a \in \mathbb{R}$ there exist at least one linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ with $\det(F) = a$.

2) Calculate the determinant of the following matrix

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 4 & 0 & -1 \\ 1 & -2 & 3 \end{pmatrix}.$$

3) We define the following elements in $\mathcal{P}_2$

$$b_1(x) = x^2 + 1, \quad b_2(x) = 2x - 1, \quad b_3(x) = 1,$$

and define the following function

$$\begin{aligned} F : \mathcal{P}_2 &\longrightarrow \mathcal{P}_2 \\ p &\longmapsto p' + 2p. \end{aligned}$$

i) Show that $B = (b_1, b_2, b_3)$ is a basis of $\mathcal{P}_2$.

ii) Show that F is a linear map.

iii) Calculate $[F]_B$.

iv) Determine the determinant of F.

4) Let $A \in \mathbb{R}^{2\times 2}$ and define

$$U_A = \{v \in \mathbb{R}^2 \mid Av = 2v\}.$$

i) Show that U_A is a subspace of $\mathbb{R}^{2\times 2}$ for any $A \in \mathbb{R}^{2\times 2}$.

ii) Find $A, B, C \in \mathbb{R}^{2\times 2}$, such that $\dim(U_A) = 0$, $\dim(U_B) = 1$, and $\dim(U_C) = 2$.

5) Let V be a finitely generated vector space. Show that a linear map $F : V \to V$ is invertible if and only if $\det(F) \neq 0$.

6) Show by induction that for all $n \geq 1$ we have

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$