

Total: 36 Points

1) (2+2+2+2=8 Points) Decide if the following statements are true or false. Justify your answers.

- i) The set $U = \{f \in C^\infty(\mathbb{R}, \mathbb{R}) \mid f'(x) = 2f(x) + x \text{ for all } x \in \mathbb{R}\}$ is a subspace of $C^\infty(\mathbb{R}, \mathbb{R})$.
- ii) The set $W = \{A \in \mathbb{R}^{2 \times 2} \mid \det(A) = 0\}$ is a subspace of $\mathbb{R}^{2 \times 2}$.
- iii) For any $A \in \mathbb{R}^{2 \times 2}$ there exists a $B \in \mathbb{R}^{2 \times 2}$ with $\det(A + B) = 0$.
- iv) If (b_1, b_2) is a basis of a vector space V , then $(b_1 - b_2, b_2 - b_1)$ is also a basis of V .

2) (4+4=8 Points) Consider the bases $B = (2x + 1, x + 1)$ and $C = (x, x - 1)$ of \mathcal{P}_1 and the linear map $F : \mathcal{P}_1 \rightarrow \mathcal{P}_1$ with

$$[F]_B^C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and $g \in \mathcal{P}_1$ with $g(x) = x$.

- i) Determine $F(g)$.
- ii) Calculate the determinant of F .

(You do not need to show that B and C are bases)

3) (5 Points) Find all $a \in \mathbb{R}$ such that the following matrix is invertible

$$M_a = \begin{pmatrix} a & -5 & 3 \\ 0 & 3 & -2 \\ -5 & 1 & 1 \end{pmatrix}.$$

4) (3+2+3+2=10 Points) We define the following elements in $\mathbb{R}^{2 \times 2}$

$$m_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, m_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, m_3 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, m_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and define with $H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ the following function

$$F : \mathbb{R}^{2 \times 2} \longrightarrow \mathbb{R}^{2 \times 2} \\ A \longmapsto HA.$$

- i) Show that $M = (m_1, m_2, m_3, m_4)$ is a basis of $\mathbb{R}^{2 \times 2}$.
- ii) Show that F is a linear map.
- iii) Calculate $[F]_M$.
- iv) Calculate the determinant of F .

5) (5 Points) Show by induction that for all $n \geq 1$ we have

$$\sum_{j=1}^n 2^{j-1} = 2^n - 1.$$

After finishing this exam please submit it at NUCT at the "Midterm" Assignment.
(Deadline: 1st June 2021 12:15 (JST))