Homework 6: Eigenvalues & Eigenvectors III

Deadline: 4th July (23:55 JST), 2021

Exercise 1. (2+4+2 = 8 Points) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

i) Calculate all eigenvalues of A.

ii) Determine orthonormal bases for all eigenspaces of A.
(Recall the Gram-Schmidt algorithm from Linear Algebra I for this)

iii) Find an orthogonal matrix $S \in \mathbb{R}^{3\times 3}$, such that $S^T AS$ is a diagonal matrix.

Exercise 2. (3+3 = 6 Points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with $A = [F]_B$, where $B = (e_1, \dots, e_n)$. Show the following two implications:

i) If $(F(e_1), \dots, F(e_N))$ is an orthonormal basis then A is invertible and $A^{-1} = A^T$.

ii) If $A^{-1} = A^T$ then $F(x) \bullet F(y) = x \bullet y$ for all $x, y \in \mathbb{R}^n$.

Exercise 3. (8 Points) Show that a matrix $A \in \mathbb{R}^{2\times 2}$ is symmetric, i.e. $A = A^T$, if and only if there exists an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

(This is the $n = 2$ case of the spectral theorem (Theorem 5.17), which you are not allowed to use to solve this Exercise. You are also not allowed to use Lemma 5.19 and need to prove it in this special case.)

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