Solution to Homework 5: Eigenvalues & Eigenvectors II

Exercise 1. (6 Points) We define the sequence $(a_n)_{n\geq 0}$ recursively by $a_0 = 0$, $a_1 = 1$ and for $n \geq 0$ by

$$a_{n+2} = 2a_n + a_{n+1}\,.$$

Determine an explicit formula for a_n by using diagonalization.
(Hint: Compare this to the Fibonacci number example in Homework 2 Exercise 1)

Solution to Exercise 1.
First we define a linear map F such that

$$\begin{aligned} F : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} &\longmapsto \begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} \end{aligned}$$

First we see that (for $E = (e_1, e_2)$)

$$\begin{pmatrix} a_{n+2} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 2a_n + a_{n+1} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} \implies A := [F]_E = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

And then, we observe

$$F \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} \implies F^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$$

We then calculate $\det A - \lambda I_2$:

$$\begin{aligned} \det(A - \lambda I_2) &= \det \begin{pmatrix} 1-\lambda & 2 \\ 1 & -\lambda \end{pmatrix} \\ &= (1-\lambda)(-\lambda) - (2)(1) \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1)\,. \end{aligned}$$

Therefore, the eigenvalues of A are

$$\lambda_1 = 2 \qquad \lambda = -1$$

For $\lambda = 2$, let $v \in \ker(A - 2I_2) = E_2(A)$. Therefore, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfies

$$\begin{aligned} \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -v_1 + 2v_2 \\ v_1 - 2v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Solving the two linear equations gives the result that

$$\ker(A - 2I_2) = E_2(A) = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

For $\lambda = -1$, let $v \in \ker(A + I_2) = E_{-1}(A)$. Therefore, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ satisfies

$$\begin{aligned} \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 2v_1 + 2v_2 \\ v_1 + v_2 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

Solving the two linear equations gives the result that

$$\ker(A + I_2) = E_{-1}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

Therefore one eigenbasis of F is $B = \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$.

We then find the matrix of F with respect to basis B by finding $[F(b_1)]_B$ and $[F(b_2)]_B$

$$\begin{aligned} [F(b_1)]_B &= [\lambda_1 b_1]_B \\ &= [2b_1]_B = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ [F(b_2)]_B &= [\lambda_2 b_2]_B \\ &= [-b_2]_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}$$

From which we infer that

$$[F]_B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

And then we observe that

$$\begin{aligned} [e_1]_B &= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_B = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix} & [e_2]_B &= \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_B = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix} \\ [b_1]_E &= \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]_E = \begin{pmatrix} 2 \\ 1 \end{pmatrix} & [b_2]_E &= \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right]_E = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

And therefore

$$S_B^E = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \qquad S_E^B = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$$

Then, we find an expression for $[F]_E^n$:

$$[F]_E = S_B^E [F]_B S_E^B \implies [F]_E^n = \left(S_B^E [F]_B S_E^B \right)^n = S_B^E [F]_B^n S_E^B$$

Which then, we substitute known matrices to find F_E^n:

$$\begin{aligned} [F]_E^n &= \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}^n \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & (-1)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2^{n+1} & (-1)^n \\ 2^n & (-1)^{n+1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2^{n+1} + (-1)^n & 2^{n+1} + (-2)(-1)^n \\ 2^n + (-1)^{n+1} & 2^n + (-2)(-1)^{n+1} \end{pmatrix} \end{aligned}$$

As $[F]_E^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$,

$$\begin{aligned} \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 2^{n+1} + (-1)^n & 2^{n+1} + (-2)(-1)^n \\ 2^n + (-1)^{n+1} & 2^n + (-2)(-1)^{n+1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2^{n+1} + (-1)^n \\ 2^n + (-1)^{n-1} \end{pmatrix} \end{aligned}$$

Which means that the explicit formula for a_n is

$$a_n = \underline{\underline{\frac{1}{3}\left(2^n + (-1)^{n-1}\right)}}\ .$$

Exercise 2. (4+4 = 8 Points)

i) Let $U \subset \mathbb{R}^n$ be a subspace and let $P_U : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection to U. Show that P_U is diagonalizable. What are the eigenvalues of P_U?
(See Linear Algebra I Section 12 & 13 for the definition of P_U)

ii) Let $\mathrm{rot}_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by an angle $\alpha \in [0, 2\pi]$. For which α is rot_α diagonalizable?

Solution to Exercise 2.

i) The map P_U is diagonalizable if there exists an eigenbasis of P_U

Let $\dim(U) = m$, $m \leq n$. By Theorem 12.6 (Linear Algebra I) there exists $B_1 = (u_1, \dots, u_m)$ that is an ONB of U.

The Orthogonal Projection to U is the map which is defined as the map

$$\begin{aligned} P_U : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto x_{||} \end{aligned}$$

We have that $\ker(P_U) = U^\perp$ is a subspace of $\mathbb{R}^n$. As $\mathrm{im}(P_U) = U$, and

$$\dim(\ker(P_U)) + \dim(\mathrm{im}(P_U)) = n$$

so $\dim(U^\perp) = n - m$. By Theorem 12.6 there exists $B_2 = (v_1, \dots, v_{n-m})$ that is an ONB of $U^\perp$. We observe that for all $x \in U^\perp$,

$$P_U(x) = (x \bullet u_1)u_1 + \cdots + (x \bullet u_m)u_m = 0$$

As for all $x \in U^\perp$, $x \bullet u = 0$ for $u \in U$. As such, <u>0 is an eigenvalue of P_U.</u>

$$E_0(P_U) = \mathrm{span}\{v_1, \dots, v_{n-m}\}\,.$$

While for all $x \in U$, we can write

$$x = \lambda_1 u_1 + \cdots + \lambda_m u_m$$

And as $B_1 = (u_1, \dots, u_m)$ is an ONB, we have

$$P_U(x) = (x \bullet u_1)u_1 + \cdots + (x \bullet u_m)u_m$$

As B_1 is an ONB, then $u_i \bullet u_j = 0$ if $i \neq j$. Thus,

$$\begin{aligned} P_U(x) &= (\lambda_1 u_1 \bullet u_1)u_1 + \dots (\lambda_m u_m \bullet u_m)u_m \\ &= \lambda_1 u_1 + \cdots + \lambda_m u_m = x \end{aligned}$$

From this we conclude that <u>1 is an eigenvalue of P_U</u> and

$$E_1(P_U) = U = \mathrm{span}\{u_1, \dots, u_m\}$$

We see that $\dim(E_1(P_U)) = m$ and $\dim(E_0(P_U)) = n - m$, so, for $\lambda_1 = 0$ and $\lambda_2 = 1$,

$$\sum_{j=1}^{2} \dim E_{\lambda_j}(P_U) = n - m + m = n$$

Then we infer that <u>P_U is diagonalizable.</u>

ii) The generic rotation matrix (by an angle α) in $\mathbb{R}^2$ is represented by

$$[\mathrm{rot}_\alpha] = A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

A linear map in $\mathbb{R}^2$ is diagonalizable if there exists an eigenbasis B with $\dim B = 2$.

Naturally, to have an eigenbasis, the linear map must have at least one eigenvalue.

Now, we find $\det(A - \lambda I_2)$:

$$\begin{aligned} \det(A - \lambda I_2) &= \det \begin{pmatrix} \cos\alpha - \lambda & -\sin\alpha \\ \sin\alpha & \cos\alpha - \lambda \end{pmatrix} \\ &= (\cos\alpha - \lambda)^2 + \sin^2\alpha \,. \end{aligned}$$

Since both terms are ≥ 0 we need to have $\sin(\alpha) = 0$, which is just the case when $\alpha = 0$, $\alpha = 2\pi$ and $\alpha = \pi$. The eigenvalues are then given by $\lambda_1 = 1 = \cos(0) = \cos(2\pi)$ and $\lambda_2 = -1 = \cos(\pi)$.

For $\alpha = 0, 2\pi$, $[\mathrm{rot}_0] = [\mathrm{rot}_{2\pi}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$, which is a diagonal matrix (thus it is diagonalizable).

For $\alpha = \pi$, $[\mathrm{rot}_\pi] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which is a diagonal matrix (thus it is diagonalizable).

Therefore, the only values of α for which rot is diagonalizable are 0, π and 2π.

Exercise 3. (4+4 = 8 Points) Let $F : V \to V$ be a linear map with eigenvalues $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and $\dim V = n$. Show the following statements by using Theorem 5.6:

i) If $B_1, \ldots, B_m$ are bases of $E_{\lambda_1}(F), \ldots, E_{\lambda_m}(F)$, then $B_1 \cup \cdots \cup B_m$ are linearly independent. (Here we mean by $B_1 \cup \cdots \cup B_m$ the collection of all vectors in the bases $B_1, \ldots, B_m$.)

ii) The map F is diagonalizable if and only if

$$\sum_{j=1}^{m} \dim E_{\lambda_j}(F) = n \,.$$

Solution to Exercise 3.

i) We write

$$\begin{aligned} B_1 &= \left(b_1^{(1)}, \ldots, b_{d_1}^{(1)}\right) \\ B_2 &= \left(b_1^{(2)}, \ldots, b_{d_2}^{(2)}\right) \\ &\cdots \\ B_m &= \left(b_1^{(m)}, \ldots, b_{d_m}^{(m)}\right) \end{aligned}$$

for the bases of the eigenspaces.

We want to show that all $b_1^{(1)}, \ldots, b_{d_1}^{(1)}, b_1^{(2)}, \ldots, b_{d_2}^{(2)}, \ldots, b_1^{(m)}, \ldots, b_{d_m}^{(m)}$ are linearly independent.

Now assume we have that

$$\sum_{i=1}^{m} \sum_{j=1}^{d_i} \alpha_{i,j} b_j^{(i)} = 0$$

We want to show that all $\alpha_{i,j} = 0$.

We define

$$u_i := \sum_{j=1}^{d_i} \alpha_{i,j} b_j^{(i)}$$

So the sum becomes

$$\sum_{i=1}^{m} u_i = 0.$$

One observes that u_i are elements in $E_{\lambda_i}(F)$, i.e. they are either the zero vector or eigenvectors to different eigenvalues. By **Theorem 5.6.** we have that eigenvectors to different eigenvalues are liearly independent so we obtain that $u_i = 0 \ \forall i$.

Then we conclude that $\alpha_{i,j} = 0$ for all i, j since for a fixed i the $b_j^{(i)}$ are linearly independent.

And thus $b_1^{(1)}, \dots, b_{d_1}^{(1)}, b_1^{(2)}, \dots, b_{d_2}^{(2)}, \dots, b_1^{(m)}, \dots, b_{d_m}^{(m)}$ are linearly independent.

ii) "$\Longrightarrow$": If F is diagonalizable, then exists an eigenbasis $B = (b_1, \dots, b_n)$.

The b_i for the same eigenvalue λ_j also forms a basis of the eigenspace $E_{\lambda_j}(F)$.

Therefore,

$$\sum_{j=1}^{m} \dim E_{\lambda_j}(F) = n \ .$$

"$\Longleftarrow$": Let B_1, B_2, $\dots$, B_m be bases of the eigenspaces $E_{\lambda_1}(F)$, $E_{\lambda_2}(F)$, $\dots$, $E_{\lambda_m}(F)$, respectively.

By the result of part i) we have that

$$B = B_1 \cup \dots \cup B_m \quad \text{linearly independent.}$$

As such, we get that

$$\dim B = \sum_{j=0}^{m} \dim \operatorname{span}\{B_j\} = \sum_{j=0}^{m} \dim E_{\lambda_j}(F)$$

If $\sum_{j=1}^{m} \dim E_{\lambda_j}(F) = n$, then that means that B is a collection of n linearly independent eigenvectors. This means that B is an eigenbasis of F.

As such, by Lemma 5.5, we conclude that F is diagonalizable.