Homework 5: Eigenvalues & Eigenvectors II

Deadline: 27th June (23:55 JST), 2021

Exercise 1. (6 Points) We define the sequence $(a_n)_{n\geq 0}$ recursivley by $a_0 = 0$, $a_1 = 1$ and for $n \geq 0$ by

$$a_{n+2} = 2a_n + a_{n+1} \,.$$

Determine an explicit formula for a_n by using diagonalization.
(Hint: Compare this to the Fibonacci number example in Homework 2 Exercise 1)

Exercise 2. (4+4 = 8 Points)

i) Let $U \subset \mathbb{R}^n$ be a subspace and let $P_U : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection to U. Show that P_U is diagonalizable. What are the eigenvalues of P_U?
(See Linear Algebra I Section 12 & 13 for the definition of P_U)

ii) Let $\mathrm{rot}_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by an angle $\alpha \in [0, 2\pi]$. For which α is rot_α diagonalizable?

Exercise 3. (4+4 = 8 Points) Let $F : V \to V$ be a linear map with eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and $\dim V = n$. Show the following statements by using Theorem 5.6:

i) If $B_1, \dots, B_m$ are bases of $E_{\lambda_1}(F), \dots, E_{\lambda_m}(F)$, then $B_1 \cup \dots \cup B_m$ are linearly independent.
(Here we mean by $B_1 \cup \dots \cup B_m$ the collection of all vectors in the bases $B_1, \dots, B_m$.)

ii) The map F is diagonalizable if and only if

$$\sum_{j=1}^{m} \dim E_{\lambda_j}(F) = n \,.$$

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