Solution to Homework 4: Eigenvalues & Eigenvectors I

Deadline: 21st June (23:55 JST), 2021

Exercise 1. (6+6 = 12 Points)

i) Consider the linear map $F : \mathbb{R}^4 \to \mathbb{R}^4$, given by $F(x) = Ax$, where

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Determine all eigenvalues of F and determine for each eigenvalue λ a basis of the eigenspace $E_\lambda(F)$.

ii) We define the linear map $G : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ for $p \in \mathcal{P}_2$ by

$$(G(p))(x) = (x^2+1)p''(x) + (x-1)p'(x) + p(x)\,.$$

Determine all eigenvalues of G and determine for each eigenvalue λ a basis of the eigenspace $E_\lambda(G)$.

Solution to Exercise 1.

i) To find the eigenvalues of F, we first consider the matrix

$$A - \lambda I_4 = \begin{pmatrix} 1-\lambda & 0 & -1 & 0 \\ 0 & 1-\lambda & 0 & 1 \\ -1 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & 1-\lambda \end{pmatrix}$$

And find its determinant (Characteristic polynomial of F) by Laplace Expansion over the 1^{st} row:

$$\det(A-\lambda I_4) = (1-\lambda)\det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix} + (-1)\det\begin{pmatrix} 0 & 1-\lambda & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

The determinant of the orange matrix can be found by Laplace Expansion over the 1^{st} row:

$$\begin{aligned} \det\begin{pmatrix} 1-\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix} &= (1-\lambda)\det\begin{pmatrix} -\lambda & 0 \\ 1 & 1-\lambda \end{pmatrix} - (0)\det\begin{pmatrix} 1 & 0 \\ 0 & 1-\lambda \end{pmatrix} + (1)\det\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \\ &= (1-\lambda)((-\lambda)(1-\lambda) - (1)(0)) + (1) \\ &= (1-\lambda)(\lambda^2-\lambda) + 1 \\ &= -\lambda^3 + 2\lambda^2 - \lambda + 1\,. \end{aligned}$$

Similarly, the determinant of the green matrix can be found by Laplace Expansion over the 3^{rd} row

$$\begin{aligned} \det\begin{pmatrix} 0 & 1-\lambda & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} &= (1-\lambda)\det\begin{pmatrix} 0 & 1-\lambda \\ -1 & 1 \end{pmatrix} \\ &= (1-\lambda)(1-\lambda)\,. \end{aligned}$$

Thus we can find the expression for $\det(A - \lambda I_4)$:

$$\begin{aligned}\det(A-\lambda I_4) &= (1-\lambda)(-\lambda^3+2\lambda^2-\lambda+1)-(1-\lambda)^2\\ &= (1-\lambda)(-\lambda^3+2\lambda^2-\lambda+1-(1-\lambda))\\ &= (1-\lambda)(-\lambda^3+2\lambda^2-\lambda+1-\lambda-1)\\ &= (1-\lambda)(-\lambda^3+2\lambda^2)\\ &= \lambda^2(1-\lambda)(-\lambda+2)\,.\end{aligned}$$

From there we conclude that the eigenvalues of F are:

$$\lambda_1 = \underline{0} \qquad \lambda_2 = \underline{1} \qquad \lambda_3 = \underline{2}$$

Let $v \in E_0(F)$. Then, $v \in \ker(A - 0I_4)$. Therefore, v must satisfy

$$\begin{pmatrix} 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 1\\ -1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}v_1 - v_3\\ v_2 + v_4\\ -v_1 + v_2\\ v_3 + v_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Solving the (equivalent) four linear equations gives the result that:

$$\ker(A) = \operatorname{span}\left\{\begin{pmatrix}-1\\-1\\-1\\1\end{pmatrix}\right\}. \text{ Therefore, } \underline{B = \left\{\begin{pmatrix}-1\\-1\\-1\\1\end{pmatrix}\right\}} \text{ is a basis of } E_0(F).$$

Let $v \in E_1(F)$. Then, $v \in \ker(A - I_4)$. Therefore, v must satisfy

$$\begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 1 & -1 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\\v_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}-v_3\\ v_4\\ -v_1 + v_2 - v_3\\ v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Solving the (equivalent) four linear equations gives the result that:

$$\ker(A - I_4) = \operatorname{span}\left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\}. \text{ Therefore, } \underline{B = \left\{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right\}} \text{ is a basis of } E_1(F).$$

Let $v \in E_2(F)$. Then, $v \in \ker(A - 2I_4)$. Then, v must satisfy

$$\begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -v_1 - v_3 \\ -v_2 + v_4 \\ -v_1 + v_2 - 2v_3 \\ v_3 - v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the (equivalent) four linear equations gives the result that:

$$\ker(A - 2I_4) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \text{ Therefore, } \underline{B = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}} \text{ is a basis of } E_2(F).$$

ii) Let us define the basis $B = (x^2, x, 1)$ as a basis of $\mathcal{P}_2$.
We observe that the Matrix of the Linear Map G with respect to B is:

$$[G]_B = \begin{pmatrix} | & | & | \\ [G(x^2)(x)]_B & [G(x)(x)]_B & [G(1)(x)]_B \\ | & | & | \end{pmatrix}$$

First, we find $[G(x^2)(x)]_B$:

$$\begin{aligned} [G(x^2)(x)]_B &= \left[(x^2+1)\frac{d^2}{dx^2}[x^2] + (x-1)\frac{d}{dx}[x^2] + x^2\right]_B \\ &= \left[(x^2+1)(2) + 2x(x-1) + x^2\right]_B \\ &= \left[5x^2 - 2x + 2\right]_B \\ &= \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}. \end{aligned}$$

Next, we find $[G(x)(x)]_B$:

$$\begin{aligned} [G(x)(x)]_B &= \left[(x^2+1)\frac{d^2}{dx^2}[x] + (x-1)\frac{d}{dx}[x] + x\right]_B \\ &= [0 + 1(x-1) + x]_B \\ &= [2x - 1]_B \\ &= \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}. \end{aligned}$$

Next, we find $[G(1)(x)]_B$:

$$\begin{aligned} [G(1)(x)]_B &= \left[(x^2+1)\frac{d^2}{dx^2}[1] + (x-1)\frac{d}{dx}[1] + 1\right]_B \\ &= [0 + 0 + 1]_B \\ &= [1]_B \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Therefore the matrix of G with respect to B is

$$[G]_B = \begin{pmatrix} 5 & 0 & 0 \\ -2 & 2 & 0 \\ 2 & -1 & 1 \end{pmatrix}$$

From there, we consider the matrix

$$[G]_B - \lambda I_3 = \begin{pmatrix} 5-\lambda & 0 & 0 \\ -2 & 2-\lambda & 0 \\ 2 & -1 & 1-\lambda \end{pmatrix}$$

And find its determinant (Characteristic Polynomial of G) with Laplace Expansion over the 1^{st} row:

$$\begin{aligned} \det\Big([G]_B - \lambda I_3\Big) &= (5-\lambda)\det\begin{pmatrix} 2-\lambda & 0 \\ -1 & 1-\lambda \end{pmatrix} - 0 + 0 \\ &= (5-\lambda)((2-\lambda)(1-\lambda) - (-1)(0)) \\ &= (5-\lambda)(2-\lambda)(1-\lambda)\,. \end{aligned}$$

Which gives that the Eigenvalues of G are:

$$\lambda_1 = \underline{5} \qquad \lambda_2 = \underline{2} \qquad \lambda_3 = \underline{1}$$

Let $v \in E_5(G)$. Then, $[v]_B \in \ker\Big([G]_B - 5I_3\Big)$. Therefore, $[v]_B$ must satisfy

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & -3 & 0 \\ 2 & -1 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ -2v_1 - 3v_2 \\ 2v_1 - v_2 - 4v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$\ker\Big([G]_B - 5I_3\Big) = \operatorname{span}\left\{\begin{pmatrix} \frac{3}{2} \\ -1 \\ 1 \end{pmatrix}\right\}$. Therefore, $\underline{B = \left\{\frac{3}{2}x^2 - x + 1\right\}}$ is a basis of $E_5(G)$.

Let $v \in E_2(G)$. Then, $[v]_B \in \ker\Big([G]_B - 2I_3\Big)$. Therefore, $[v]_B$ must satisfy

$$\begin{pmatrix} 3 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 3v_1 \\ -2v_1 \\ 2v_1 - v_2 - v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$\ker\Big([G]_B - 2I_3\Big) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$. Therefore, $\underline{B = \{x - 1\}}$ is a basis of $E_2(G)$.

Let $v \in E_1(G)$. Then, $[v]_B \in \ker\Big([G]_B - I_3\Big)$. Therefore, $[v]_B$ must satisfy

$$\begin{pmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4v_1 \\ -2v_1 + v_2 \\ 2v_1 - v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$$\ker\Big([G]_B - I_3\Big) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}.$$ Therefore, $B = \{1\}$ is a basis of $E_1(G)$.

Remark: For $F : V \longrightarrow V$, and $\dim(V) = n$, $E_\lambda(F) \subseteq V$ while $\ker([F]_B - \lambda I_n) \subseteq \mathbb{R}^n$. So if one asks for the eigenvectors of F the answer should be elements in V.

Generally, to find the eigenvalues of a linear map $F : V \to V$, we must consider a basis B of the space V and then calculate the coordinate vectors of the eigenvector(s) of F.

However, if $V = \mathbb{R}^n$, we use the standard basis $E = (e_1, \ldots, e_n)$, so we can treat the coordinate vectors as elements of V.

Exercise 2. (2+2+2+2 = 8 Points) Give examples of linear maps $F_i : \mathbb{R}^3 \to \mathbb{R}^3$ for $i = 1, 2, 3, 4$, such that

i) F_1 has exactly three different eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

ii) F_2 has exactly two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\dim E_{\lambda_1}(F_2) = \dim E_{\lambda_2}(F_2) = 1$.

iii) F_3 has exactly two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\dim E_{\lambda_1}(F_3) = 1$ and $\dim E_{\lambda_2}(F_3) = 2$.

iv) F_4 has exactly one eigenvalue $\lambda_1 \in \mathbb{R}$.

In each case calculate a basis for the eigenspaces.

Solution to Exercise 2.

i) We define the linear map F_1 as:

$$F_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$x \longmapsto Ax \text{ where } A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This linear map has three eigenvalues, namely

$$\lambda_1 = -1 \qquad \lambda_2 = 0 \qquad \lambda_3 = 1$$

Let $v \in E_{-1}(F_1)$. Then, $v \in \ker\big(A + I_3\big)$. Therefore, v must satisfy

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ v_2 \\ 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$$\ker\left(A + I_3\right) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}. \text{ Therefore, } \underline{B = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}} \text{ is a basis of } E_{-1}(F_1).$$

Let $v \in E_0(F_1)$. Then, $v \in \ker\left(A\right)$. Therefore, v must satisfy

$$\begin{pmatrix}-1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$
$$\begin{pmatrix}-v_1\\0\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$$\ker\left(A\right) = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}. \text{ Therefore, } \underline{B = \left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}} \text{ is a basis of } E_0(F_1).$$

Let $v \in E_1(F_1)$. Then, $v \in \ker\left(A - I_3\right)$. Therefore, v must satisfy

$$\begin{pmatrix}-2 & 0 & 0\\0 & -1 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$
$$\begin{pmatrix}-2v_1\\-v_2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$$\ker\left(A - I_3\right) = \text{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}. \text{ Therefore, } \underline{B = \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}} \text{ is a basis of } E_1(F_1).$$

ii) We define the linear map F_2 as:

$$F_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$x \longmapsto Cx \text{ where } C = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}.$$

This linear map has two eigenvalues, namely

$$\lambda_1 = \underline{0} \qquad \lambda_2 = \underline{1}$$

Let $v \in E_0(F_2)$. Then, $v \in \ker\left(C\right)$. Therefore, v must satisfy

$$\begin{pmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$
$$\begin{pmatrix}v_1 + v_2 + v_3\\v_2 + v_3\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$$\ker\big(C\big) = \text{span}\left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}. \text{ Therefore, } \underline{B = \left\{\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}} \text{ is a basis of } E_0(F_2).$$

Let $v \in E_1(F_2)$. Then, $v \in \ker\big(C - I_3\big)$. Therefore, v must satisfy

$$\begin{pmatrix}0 & 1 & 1\\0 & 0 & 1\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}v_2+v_3\\v_3\\-v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$$\ker\big(C - I_3\big) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}. \text{ Therefore, } \underline{B = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}} \text{ is a basis of } E_1(F_2).$$

iii) We define the linear map F_3 as:

$$F_3 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$x \longmapsto Dx \text{ where } D = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}.$$

This linear map has two eigenvalues, namely

$$\lambda_1 = \underline{0} \qquad \lambda_2 = \underline{1}$$

Let $v \in E_0(F_3)$. Then, $v \in \ker\big(D\big)$. Therefore, v must satisfy

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}v_1\\v_2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$$ker\big(D\big) = \text{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}. \text{ Therefore, } \underline{B = \left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}} \text{ is a basis of } E_0(F_3).$$

Let $v \in E_1(F_3)$. Then, $v \in \ker\big(D - I_3\big)$. Therefore, v must satisfy

$$\begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\-v_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving the (equivalent) three linear equations gives the result that:

$$\ker\left(D - I_3\right) = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}. \text{ Therefore, } \underline{B = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}} \text{ is a basis of } E_1(F_3).$$

iv) We define the linear map F_4 as:

$$\begin{aligned} F_4 : \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ x &\longmapsto I_3 x \text{ where } I_3 = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}. \end{aligned}$$

This linear map has exactly eigenvalues, namely

$$\lambda_1 = \underline{1}$$

Observe that for all $x \in \mathbb{R}^3$, $F_4(x) = I_3 x = 1 \cdot x$, so $E_1(F_4) = \mathbb{R}^3$, and thus the standard basis of $\mathbb{R}^3$

$$\underline{B = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}} \text{ is a basis of } E_1(F_4).$$

Exercise 3. (4 Points) Let V be a finitely generated vector-space. Show that a linear map $F : V \to V$ is invertible if and only if 0 is not an eigenvalue of F.

Solution to Exercise 3.

For this exercise, we assume that $\dim(V) = n$ for an arbitrary $n \in \mathbb{N}$.

"$\Longrightarrow$": If $F : V \to V$ is invertible, then 0 is not an eigenvalue of F.

Suppose there exists an invertible linear map F with 0 as one of its eigenvalues.
Then there exists $v \in V$ with $v \neq n_V$ such that $F(x) = 0 \cdot v = n_V$. Therefore,

$$v \in \ker(F) \neq \{n_V\}$$

Where n_V is the neutral element of the vector space V.
As such, $\ker(F) \neq \{n_V\}$, contradicting the fact that F is invertible.
Therefore, there exists no such F, proving the statement. (Proof by Contradiction).

"$\Longleftarrow$": If 0 is not an eigenvalue of F, then F is invertible.

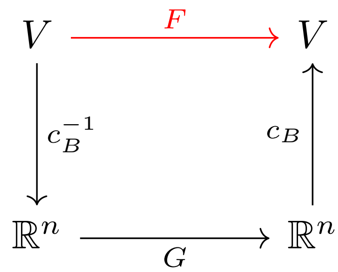
As V is a finitely generated vector space, we can always find B, an arbitrary basis of V.
As 0 is not an eigenvalue of F, we have that

$$\det(F) = \det([F]_B) \neq 0$$

Which implies that the matrix $[F]_B$ is invertible.
We then define two linear maps (due to the fact that $\dim(V) = n$):

$$\begin{aligned} G : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ v &\longmapsto [F]_B v \end{aligned} \qquad \begin{aligned} H : V &\longrightarrow V \\ v &\longmapsto [F]_B^{-1} v \end{aligned}$$

This Figure shows the relationship between F, G, the coordinate map (and its inverse) c_B and c_B^{-1}:



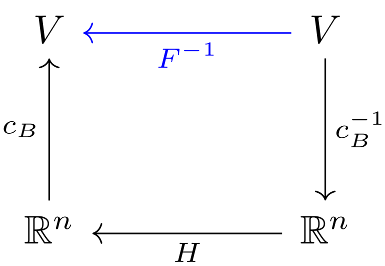
From this diagram we observe that

$$F(v) = (c_B \circ G \circ c_B^{-1})(v)$$

Next, we define the map

$$\begin{aligned} F^{-1} : V &\longrightarrow V \\ v &\longmapsto (c_B \circ H \circ c_B^{-1})(v) \end{aligned}$$

This Figure shows the relationship between F^{-1}, H, the coordinate map (and its inverse) c_B and c_B^{-1}:



Then, we observe that:

$$[G \circ H] = [F]_B[F]_B^{-1} = I_n \iff G \circ H = \mathrm{id}$$

Now, we consider the expression $(F \circ F^{-1})(v)$ for an arbitrary $v \in V$:

$$(F \circ F^{-1})(v) = (c_B \circ G \circ c_B^{-1} \circ c_B \circ H \circ c_B^{-1})(v)$$

By definition, $c_B \circ c_B^{-1} = \mathrm{id}$:

$$\begin{aligned} (F \circ F^{-1})(v) &= (c_B \circ G \circ \mathrm{id} \circ H \circ c_B^{-1})(v) \\ &= (c_B \circ G \circ H \circ c_B^{-1})(v) \\ &= (c_B \circ \mathrm{id} \circ c_B^{-1})(v) \\ &= (c_B \circ c_B^{-1})(v) \\ &= \mathrm{id}(v)\,. \end{aligned}$$

Which proves that F^{-1} is indeed the inverse of F, and that F is invertible, proving the statement.