Homework 4: Eigenvalues & Eigenvectors I

Deadline: 19th June (23:55 JST), 2021

Exercise 1. (6+6 = 12 Points)

i) Consider the linear map $F : \mathbb{R}^4 \to \mathbb{R}^4$, given by $F(x) = Ax$, where

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Determine all eigenvalues of F and determine for each eigenvalue λ a basis of the eigenspace $E_\lambda(F)$.

ii) We define the linear map $G : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$ for $p \in \mathcal{P}_2$ by

$$(G(p))(x) = (x^2+1)p''(x) + (x-1)p'(x) + p(x)\,.$$

Determine all eigenvalues of G and determine for each eigenvalue λ a basis of the eigenspace $E_\lambda(G)$.

Exercise 2. (2+2+2+2 = 8 Points) Give examples of linear maps $F_i : \mathbb{R}^3 \to \mathbb{R}^3$ for $i = 1,2,3,4$, such that

i) F_1 has exactly three different eigenvalues $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

ii) F_2 has exactly two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\dim E_{\lambda_1}(F_2) = \dim E_{\lambda_2}(F_2) = 1$.

iii) F_3 has exactly two different eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\dim E_{\lambda_1}(F_3) = 1$ and $\dim E_{\lambda_2}(F_3) = 2$.

iv) F_4 has exactly one eigenvalue $\lambda_1 \in \mathbb{R}$.

In each case calculate a basis for the eigenspaces.

Exercise 3. (4 Points) Let V be a finitely generated vector-space. Show that a linear map $F : V \to V$ is invertible if and only if 0 is not an eigenvalue of F.