Solution to Homework 2: Matrix of a linear map & Induction

Exercise 1. (1+3+3+3 = 10 Points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}\,. \qquad (n \geq 2)$$

In this exercise we want to prove the following explicit formula

$$F_n = \frac{1}{2^n\sqrt{5}}\left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right)\,. \tag{0.1}$$

For this follow the following steps:

i) Find a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$, such that $F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for $n \geq 1$, where $F^n = \underbrace{F \circ \cdots \circ F}_{n}$.

ii) We define the following two bases of $\mathbb{R}^2$:

$$B_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right), \quad B_2 = \left(\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}\right).$$

Determine the change-of-basis matrices $S_{B_1}^{B_2}$ and $\left(S_{B_1}^{B_2}\right)^{-1}$.

iii) Calculate $[F]_{B_1}$ and $[F]_{B_2}$.

iv) Calculate $[F]_{B_1}^n$ by using

$$[F]_{B_1} = \left(S_{B_1}^{B_2}\right)^{-1} [F]_{B_2}\, S_{B_1}^{B_2}$$

and prove (0.1) by using i).

Solution to Exercise 1.

i) First, one sets $n = 1$ to find:

$$F\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

And rearranging the equation when $n = n$ to find:

$$F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$$
$$F\left(F^{n-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$$

Observe that $F_{n+1} = F_n + F_{n-1}$ and that $F^{n-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix}$, one has:

$$F\begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n-1} + F_n \end{pmatrix}$$

Then, the linear map F is:

$$\begin{aligned} F : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} y \\ x+y \end{pmatrix}. \end{aligned}$$

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ii) Henceforth, let id_V denote the identity function on the vector space V.

Let B_1 be expressed as $B_1 = (\, b_{1,1} \, , \, b_{1,2} \,)$ where $b_{1,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $b_{1,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let B_2 be expressed as $B_2 = (\, b_{2,1} \, , \, b_{2,2} \,)$ where $b_{2,1} = \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}$ and $b_{2,2} = \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}$.

By the definition of the Change of Basis Matrix, we have that:

$$S_{B_1}^{B_2} = \left[(c_{B_2})^{-1} \circ \mathrm{id}_{\mathbb{R}^2} \circ c_{B_1} \right] = \begin{pmatrix} | & | \\ [b_{1,1}]_{B_2} & [b_{1,2}]_{B_2} \\ | & | \end{pmatrix}$$

One then observes that $b_{1,1}$ and $b_{1,2}$ can be written as:

$$b_{1,1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left(\frac{1}{4\sqrt{5}}(\sqrt{5}-1) \right) b_{2,1} + \left(\frac{1}{4\sqrt{5}}(\sqrt{5}+1) \right) b_{2,2}$$

and

$$b_{1,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left(\frac{1}{2\sqrt{5}} \right) b_{2,1} + \left(-\frac{1}{2\sqrt{5}} \right) b_{2,2}$$

Therefore, we can write the change of basis matrix as:

$$S_{B_1}^{B_2} = \begin{pmatrix} \frac{\sqrt{5}-1}{4\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{\sqrt{5}+1}{4\sqrt{5}} & -\frac{1}{2\sqrt{5}} \end{pmatrix} . \quad \text{Its Inverse is: } \left(S_{B_1}^{B_2} \right)^{-1} = \begin{pmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix} .$$

iii) One has that:

$$F(b_{1,1}) = F \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \; ; [\, F(b_{1,1})]_{B_1} = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]_{B_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

$$F(b_{1,2}) = F \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \; ; [\, F(b_{1,2})]_{B_1} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_{B_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Thus, we can write

$$[F]_{B_1} = \begin{pmatrix} | & | \\ [F(b_{1,1}]_{B_1} & [F_{b_{1,2}}]_{B_1} \\ | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Next, one has that:

$$F(b_{2,1}) = F \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1+\sqrt{5} \\ 3+\sqrt{5} \end{pmatrix} = \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right) \cdot 2 \\ \left(\frac{1+\sqrt{5}}{2}\right) \cdot \frac{1+\sqrt{5}}{2} \end{pmatrix} \; ; [\, F(b_{2,1})]_{B_2} = \left[\begin{pmatrix} 1+\sqrt{5} \\ 3+\sqrt{5} \end{pmatrix} \right]_{B_2} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 0 \end{pmatrix} .$$

$$F(b_{2,2}) = F \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1-\sqrt{5} \\ 3-\sqrt{5} \end{pmatrix} = \begin{pmatrix} \left(\frac{1-\sqrt{5}}{2}\right) \cdot 2 \\ \left(\frac{1-\sqrt{5}}{2}\right) \cdot \frac{1-\sqrt{5}}{2} \end{pmatrix} \; ; [\, F(b_{2,2})]_{B_2} = \left[\begin{pmatrix} 1-\sqrt{5} \\ 3-\sqrt{5} \end{pmatrix} \right]_{B_2} = \begin{pmatrix} 0 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} .$$

Then, we have that

$$[F]_{B_2} = \begin{pmatrix} | & | \\ [F(b_{2,1}]_{B_2} & [F_{b_{2,2}}]_{B_2} \\ | & | \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

iv) One has that $[F]_{B_1} = \left(S_{B_1}^{B_2} \right)^{-1} [F]_{B_2} \, S_{B_1}^{B_2}$.

One observes that (by the definition of the inverse) : $S_{B_1}^{B_2} \left(S_{B_1}^{B_2} \right)^{-1} = [\mathrm{id}_{\mathbb{R}^2}] = I_2$.

Thus, one has that:

$$[F]_{B_1}^n = \left(S_{B_1}^{B_2} \right)^{-1} [F]_{B_2} \, S_{B_1}^{B_2} \left(S_{B_1}^{B_2} \right)^{-1} \ldots \; [F]_{B_2} \, S_{B_1}^{B_2} = \left(S_{B_1}^{B_2} \right)^{-1} [F]_{B_2}^n \, S_{B_1}^{B_2}$$

Therefore, one can calculate the matrix $[F]^n_{B_2}$ by:

$$[F]^n_{B_1} = \begin{pmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^n \cdot \begin{pmatrix} \frac{\sqrt{5}-1}{4\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{\sqrt{5}+1}{4\sqrt{5}} & -\frac{1}{2\sqrt{5}} \end{pmatrix}$$

$$[F]^n_{B_1} = \begin{pmatrix} 2 & 2 \\ 1+\sqrt{5} & 1-\sqrt{5} \end{pmatrix} \cdot \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{5}-1}{4\sqrt{5}} & \frac{1}{2\sqrt{5}} \\ \frac{\sqrt{5}+1}{4\sqrt{5}} & -\frac{1}{2\sqrt{5}} \end{pmatrix}$$

$$[F]^n_{B_1} = \begin{pmatrix} \frac{1}{2^{n-1}}(1+\sqrt{5})^n & \frac{1}{2^{n-1}}(1-\sqrt{5})^n \\ \frac{1}{2^n}(1+\sqrt{5})^{n+1} & \frac{1}{2^n}(1-\sqrt{5})^{n+1} \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{5}-1}{4\sqrt{5}} & \frac{2}{4\sqrt{5}} \\ \frac{\sqrt{5}+1}{4\sqrt{5}} & -\frac{2}{4\sqrt{5}} \end{pmatrix}$$

$$[F]^n_{B_1} = \begin{pmatrix} \frac{1}{2^{n-1}} \cdot \frac{1}{\sqrt{5}} \left((1+\sqrt{5})^{n-1} + (1-\sqrt{5})^{n-1}\right) & \frac{1}{2^n} \cdot \frac{1}{\sqrt{5}} \left((1+\sqrt{5})^n + (1-\sqrt{5})^n\right) \\ \frac{1}{2^n} \cdot \frac{1}{\sqrt{5}} \left((1+\sqrt{5})^n + (1-\sqrt{5})^n\right) & \frac{1}{2^{n+1}} \cdot \frac{1}{\sqrt{5}} \left((1+\sqrt{5})^{n+1} + (1-\sqrt{5})^{n+1}\right) \end{pmatrix}$$

By definition of the linear map we have that $[F]^n_{B_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$. Therefore,

$$[F]^n_{B_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2^n} \cdot \frac{1}{\sqrt{5}} \left((1+\sqrt{5})^n + (1-\sqrt{5})^n\right) \\ \frac{1}{2^{n+1}} \cdot \frac{1}{\sqrt{5}} \left((1+\sqrt{5})^{n+1} + (1-\sqrt{5})^{n+1}\right) \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$$

This proves Equation (0.1), namely:

$$F_n = \frac{1}{2^n\sqrt{5}} \left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right) .$$

Exercise 2. (3+3+3+3 = 12 Points) Use mathematical induction to prove the following statements.

i) For the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and all $n \geq 1$ we have

$$A^n = \begin{pmatrix} 1 & n & \binom{n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix} ,$$

where $\binom{n}{2} = \frac{n(n-1)}{2}$.
(Bonus: Can you give a analogue statement for a 4×4 or $k \times k$ version of the matrix A?)

ii) For all $n \geq 1$ we have

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 .$$

iii) The number of permutations of $\{1, \ldots, n\}$ is given by $n!$.
(Here $n! = 1 \cdot 2 \cdots n$ denotes the factorial.)

iv) Let V be a vector space which is not finitely generated. Then for any $n \geq 1$ there exist vectors $v_1, \ldots, v_n \in V$ which are linearly independent.

Solution to Exercise 2.

i) **Base Step:** For $n = 1$, we have that:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{(1)(1-1)}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \binom{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Assumption: For a fixed $n \in \mathbb{N}$, and for all $m < n$, $m \in \mathbb{N}$, the equation

$$A^m = \begin{pmatrix} 1 & m & \binom{m}{2} \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix} \quad \text{is true.}$$

Inductive Step:

$$\begin{aligned} A^n &= A^{n-1} \cdot A \\ A^n &= \begin{pmatrix} 1 & n-1 & \binom{n-1}{2} \\ 0 & 1 & n-1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Observe that $n - 1 = \binom{n-1}{1}$.

$$A^n = \begin{pmatrix} 1 & n & \binom{n-1}{2} + \binom{n-1}{1} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

Claim: For all $n \in \mathbb{N}$, we have

$$\binom{n-1}{2} + (n-1) = \binom{n}{2}$$

By the formula $\binom{n}{2} = \frac{n(n-1)}{2}$, we have

$$\begin{aligned} \binom{n-1}{2} + (n-1) &= \frac{(n-1)(n-2)}{2} + (n-1) \\ \binom{n-1}{2} + (n-1) &= \frac{n^2 - 3n + 2 + 2n - 2}{2} \\ \binom{n-1}{2} + (n-1) &= \frac{n^2 - n}{2} \\ \binom{n-1}{2} + (n-1) &= \frac{n(n-1)}{2} \\ \binom{n-1}{2} + (n-1) &= \binom{n}{2} \end{aligned}$$

Using the claim proven above, we have that

$$A^n = \begin{pmatrix} 1 & n & \binom{n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

Which proves the statement by mathematical induction.

The analogue statement for a $k \times k$ matrix (denoted as B) is that for $n \geq 1$, $n \in \mathbb{N}$,

$$B^n = \begin{pmatrix} 1 & \binom{n}{1} & \dots & \binom{n}{k-1} & \binom{n}{k} \\ 0 & 1 & & \binom{n}{k-2} & \binom{n}{k-1} \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & \binom{n}{1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Bonus Base Case (Just for fun!): For n=1, we have that:

$$B = \begin{pmatrix} 1 & 1 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \binom{1}{1} & \dots & \binom{1}{k-1} & \binom{1}{k} \\ 0 & 1 & & \binom{1}{k-2} & \binom{1}{k-1} \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & \binom{1}{1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Then, for a fixed $n \in \mathbb{N}$ one assumes that the analogue statement is true for all $m \in \mathbb{N}$, $m < n$.
Inductive Step:

$$B^n = B^{n-1} \cdot B$$

$$B^n = \begin{pmatrix} 1 & \binom{n-1}{1} & \dots & \binom{n-1}{k-1} & \binom{n-1}{k} \\ 0 & 1 & & \binom{n-1}{k-2} & \binom{n-1}{k-1} \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & \binom{n-1}{1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$B^n = \begin{pmatrix} 1 & \binom{n-1}{1}+1 & \dots & \binom{n-1}{k-1}+\binom{n-1}{k-2} & \binom{n-1}{k}+\binom{n-1}{k-1} \\ 0 & 1 & & \binom{n-1}{k-2}+\binom{n-1}{k-3} & \binom{n-1}{k-1}+\binom{n-1}{k-2} \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & \binom{n-1}{1}+1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Claim: For all $n \in \mathbb{N}$ and $k \in \mathbb{N}$, we have

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

By the formula for (the number of) combinations, we have:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$
$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{n!}{(k-1)!(n-k-1)!}\left(\frac{1}{k} + \frac{1}{n-k}\right)$$
$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)! \cdot (n)}{(k-1)! \cdot k \cdot (n-k) \cdot (n-k-1)!}$$
$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{n!}{k!(n-k)!}$$
$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

Using the claim proven above, we have that

$$B^n = \begin{pmatrix} 1 & \binom{n}{1} & \dots & \binom{n}{k-1} & \binom{n}{k} \\ 0 & 1 & & \binom{n}{k-2} & \binom{n}{k-1} \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & \binom{n}{1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Which proves the analogue statement for $k \times k$ matrices by mathematical induction.

ii) **Base Case:** for $n = 1$, one has that

$$1^3 = 1^2$$

Which proves the base case.

Assumption: for a fixed $n \in \mathbb{N}$, assume that for all $m \in \mathbb{N}$, $m < n$,

$$1^3 + 2^3 + 3^3 + \cdots + m^3 = (1 + 2 + 3 + \cdots + m)^2$$

Inductive Case:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3$$

By the assumption above:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + n^3 &= (1 + 2 + 3 + \cdots + (n-1))^2 + n^3 \\
1^3 + 2^3 + 3^3 + \cdots + n^3 &= (1 + 2 + 3 + \cdots + (n-1))^2 + n^2 + (n-1)n^2 \\
1^3 + 2^3 + 3^3 + \cdots + n^3 &= (1 + 2 + 3 + \cdots + (n-1))^2 + 2 \cdot \frac{(n-1)n^2}{2} + n^2
\end{aligned}$$

Claim: For all $n \in \mathbb{N}$, $(1 + 2 + \cdots + (n-1)\,) = \frac{(n-1)n}{2}$.
This can be proven by induction.
Base Case: For $n = 1$, we have

$$1 = \frac{1(2)}{2} = 1$$

Proving the base case.
Assumption: for a fixed $n \in \mathbb{N}$ and for all $m \in \mathbb{N}$, $m < n$,

$$1 + 2 + 3 + \cdots + m = \frac{m(m+1)}{2}$$

Inductive Step:

$$\begin{aligned}
1 + 2 + 3 + \cdots + n &= (1 + 2 + 3 + \cdots + (n-1)\,) + n \\
1 + 2 + 3 + \cdots + n &= \frac{(n-1)n)}{2} + n \\
1 + 2 + 3 + \cdots + n &= \frac{n^2 - n + 2n}{2} \\
1 + 2 + 3 + \cdots + n &= \frac{n^2 + n}{2} \\
1 + 2 + 3 + \cdots + n &= \frac{n(n+1)}{2}
\end{aligned}$$

Which proves the statement by mathematical induction.

By this statement, observe that:

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + n^3 &= (1 + 2 + 3 + \cdots + (n-1)\,)^2 + 2 \cdot (1 + 2 + 3 + \cdots + (n-1)\,)(n) + n^2 \\
1^3 + 2^3 + 3^3 + \cdots + n^3 &= (1 + 2 + 3 + \cdots + n\,)^2
\end{aligned}$$

Which proves the statement by mathematical induction.

iii) **Base Case:** for $n = 1$, there is only $1 = 1!$ possible permutation, namely $\sigma(1) = 1$.

Assumption:
For a fixed $n \in \mathbb{N}$, for all $m \in \mathbb{N}$, $m < n$, the number of permutations of m elements is given by $m!$.

Inductive Case:
To take a permutation of n elements, we can separate the process into two steps:

(a) Take the permutation of the first $n - 1$ elements. As there are n possible elements, there will be one leftover element. Let the leftover element be j.
(b) Then, set $\sigma(n) = j$ (i.e. set the permutation of the last element to be the leftover element).

One observes that there are n possibilities of j (there are n ways of going about step (b)), while (by Inductive Assumption) there are $(n - 1)!$ permutations of $n - 1$ elements. Therefore,

$$\text{Number of Permutations} = (n - 1)! \cdot (n) = n!$$

Which proves the statement by mathematical induction.

iv) Henceforth, let $\mathbf{0}$ denote the zero vector in the vector space V.

Base Case: All vectors $v_1 \in V$ is linearly independent whenever $v_1 \neq \mathbf{0}$, proving the base case.

This is because V is not the zero vector space, because the zero vector space is finitely generated (by the zero vector).

Assumption: For a fixed $n \in \mathbb{N}$, and all $m \in \mathbb{N}$, $m < n$, there exists m vectors $v_1, \ldots, v_m \in V$ which are linearly independent.
Inductive Step:

Claim: There exists a vector $v_n \in V$ such that $v_n \notin \text{span}\{v_1, ..., v_{n-1}\}$.
Proof By Contradiction.
Suppose that one can't find such v_n. If so, then one has that:

$$V = \text{span}\{\, v_1, \ldots, v_n \,\}$$

Which contradicts the statement that V is not a finitely generated vector space.
Therefore, one ascertains the existence of v_n, proving the claim.

Now consider the equation

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0\,.$$

If $\lambda_n \neq 0$, we have that:

$$-\frac{\lambda_1}{\lambda_n} v_1 + \cdots + -\frac{\lambda_{n-1}}{\lambda_n} v_{n-1} = v_n$$

Which contradicts the statement that $v_n \notin \text{span}\{v_1, \ldots, v_{n-1}\}$. Therefore, $\lambda_n = 0$.

$$\lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1} = \; 0$$

Which implies that $\lambda_1 = \cdots = \lambda_{n-1}$ as $v_1, \ldots, v_{n-1}$ are linearly independent by assumption.

Therefore, the only solution to the equation is $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.
As such, one concludes that $v_1, \ldots, v_n$ are linearly independent.

As such, the statement is proven by mathematical induction.