Homework 2: Matrix of a linear map & Induction

Deadline: 23rd May (23:55 JST), 2021

Exercise 1. (1+3+3+3 = 10 Points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}\,. \qquad (n \geq 2)$$

In this exercise we want to prove the following explicit formula

$$F_n = \frac{1}{2^n\sqrt{5}}\left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right)\,. \tag{0.1}$$

For this follow the following steps:

i) Find a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$, such that $F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for $n \geq 1$, where $F^n = \underbrace{F \circ \cdots \circ F}_{n}$.

ii) We define the following two bases of $\mathbb{R}^2$:

$$B_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\,, \quad B_2 = \left(\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}\right)\,.$$

Determine the change-of-basis matrices $S_{B_1}^{B_2}$ and $\left(S_{B_1}^{B_2}\right)^{-1}$.

iii) Calculate $[F]_{B_1}$ and $[F]_{B_2}$.

iv) Calculate $[F]_{B_1}^n$ by using

$$[F]_{B_1} = \left(S_{B_1}^{B_2}\right)^{-1} [F]_{B_2}\, S_{B_1}^{B_2}$$

and prove (0.1) by using i).

Exercise 2. (3+3+3+3 = 12 Points) Use mathematical induction to prove the following statements.

i) For the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and all $n \geq 1$ we have

$$A^n = \begin{pmatrix} 1 & n & \binom{n}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}\,,$$

where $\binom{n}{2} = \frac{n(n-1)}{2}$.
(Bonus: Can you give a analogue statement for a 4×4 or $k \times k$ version of the matrix A?)

ii) For all $n \geq 1$ we have

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2\,.$$

iii) The number of permutations of $\{1, \ldots, n\}$ is given by $n!$.
(Here $n! = 1 \cdot 2 \cdots n$ denotes the factorial.)

iv) Let V be a vector space which is <u>not</u> finitely generated. Then for any $n \geq 1$ there exist vectors $v_1, \ldots, v_n \in V$ which are linearly independent.

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