Solution to Homework 1: Vector spaces

Exercise 1. (4+2+2 = 8 Points) Let $V = \{x \in \mathbb{R} \mid x > 0\}$ be the set of all positive real numbers. Define on V the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$u \oplus v = uv \qquad \text{(the usual multiplication of real numbers)}$$
$$\lambda \odot v = v^\lambda .$$

i) Show that $(V, \oplus, \odot)$ is a vector space.
(i.e. check that the operations $\oplus$ and $\odot$ satisfy the properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$.)

ii) Determine all subspaces of $(V, \oplus, \odot)$.

iii) Find an isomorphism
$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot) .$$
Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

Solution to Exercise 1.

i) First notice that $\oplus$ and $\odot$ are actually well-defined on V, i.e. for all $u, v \in V, \lambda \in \mathbb{R}$ we have $u \oplus v, \lambda \odot v \in V$. (This you do not need to mention)

(A.1) $\forall u, v, w \in V$: $(u \oplus v) \oplus w = u \oplus (v \oplus w)$.
$(u \oplus v) \oplus w = (u \cdot v) \cdot w = u \cdot (v \cdot w) = u \oplus (v \oplus w)$.
(A.2) $\forall u, v \in V$: $u \oplus v = v \oplus u$.
$u \oplus v = u \cdot v = v \cdot u = v \oplus u$.
(A.3) $\exists n \in V$, $\forall u \in V$: $n \oplus u = u$.
Consider $n = 1 \in V$. One observes that, for all $u \in V$, $n \oplus u = 1 \cdot u = u$.
(A.4) $\forall u \in V$, $\exists v \in V$: $u \oplus v = n$.
For a fixed $u \in V$, consider $v = \frac{1}{u} \in V$ (which is always defined as $0 \notin V$). $u \oplus v = u \cdot \frac{1}{u} = 1 = n$.
(C.1) $\forall u, v \in V$, $\lambda \in \mathbb{R}$: $\lambda \odot (u \oplus v) = (\lambda \odot u) \oplus (\lambda \odot v)$.
$\lambda \odot (u \oplus v) = \lambda \odot (u \cdot v) = (uv)^\lambda = u^\lambda \cdot v^\lambda = (\lambda \odot u) \cdot (\lambda \odot v) = (\lambda \odot u) \oplus (\lambda \odot v)$.
(C.2) $\forall u \in V$, $\lambda, \mu \in \mathbb{R}$: $(\lambda + \mu) \odot u = (\lambda \odot u) \oplus (\mu \odot u)$.
$(\lambda + \mu) \odot u = u^{\lambda+\mu} = u^\lambda \cdot u^\mu = u^\lambda \oplus u^\mu = (\lambda \odot u) \oplus (\mu \odot u)$.
(C.3) $\forall u \in V$, $\lambda, \mu \in \mathbb{R}$: $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.
$\lambda \odot (\mu \odot u) = \lambda \odot (u^\mu) = (u^\mu)^\lambda = u^{\lambda\mu} = (\lambda \cdot \mu) \odot u$.
(C.4) $\forall u \in V$: $1 \odot u = u$.
It follows from the definition that $1 \odot u = u^1 = u$.

ii) The subset containing only the neutral element is always a subspace. ($\{n\} = \{1\} \subseteq V$ is a subspace). Also, the whole space V itself is also a subspace ($V \subseteq V$ is a subspace).

Claim: There exists no subspace of V other than the two mentioned above.
Suppose that $U = \{1, u_1, ..., u_n\}$ is a subspace of V with $U \subset V$.
Therefore, $\exists\ a \in V$ such that $a \notin U$. As $u_1, a \in V$, one has that
$$a = u_1^{\log_{u_1}(a)} = (\log_{u_1}(a)) \odot u_1 .$$
Here, $\log_{u_1}(a)$ is a real constant that is always defined as $u_1, a \in V$.
Thus, U is not closed under the scalar multiplication $\odot$. As such, U is not a subspace of V.

Therefore, the only subspaces of $(V, \oplus, \odot)$ are $\{1\}$ and V.

iii) The function F must fulfill the following three conditions (for $x, y, \lambda \in \mathbb{R}$):

$$\begin{aligned} F(x+y) &= F(x) \oplus F(y) = F(x) \cdot F(y) \\ F(\lambda \cdot x) &= \lambda \odot F(x) = (\, F(x)\,)^{\lambda} \\ F(x) &> 0 \quad \text{for all } x \in \mathbb{R} \end{aligned}$$

One function that comes to mind is the exponential function, defined by $\exp(x) = e^x$. In fact,

$$\begin{aligned} \exp(x+y) &= e^x \cdot e^y = \exp(x) \oplus \exp(y) \\ \exp(\lambda \cdot x) &= e^{\lambda x} = (e^x)^{\lambda} = \lambda \odot \exp(x) \\ \exp(x) &> 0 \quad \text{for all } x \in \mathbb{R} \\ \exp(x) & \quad \text{is a bijective map from } \mathbb{R} \text{ to } V. \end{aligned}$$

An isomorphism is an invertible linear map; the bijectivity of a function implies its invertibility. Therefore, an example of an isomorphism from $(\mathbb{R}, +, \cdot)$ to $(V, \oplus, \odot)$ would be:

$$\begin{aligned} F : (\mathbb{R}, +, \cdot) &\longrightarrow (V, \oplus, \odot) \\ x &\longmapsto e^x \end{aligned}$$

In fact, one can observe that all functions with the form:

$$\begin{aligned} G : (\mathbb{R}, +, \cdot) &\longrightarrow (V, \oplus, \odot) \\ x &\longmapsto k^x\,, \;\; k \in V \setminus \{1\} \end{aligned}$$

is an isomorphism from $(\mathbb{R}, +, \cdot)$ to $(V, \oplus, \odot)$.

Exercise 2. (4+2+2 = 8 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \left\{ A \in \mathbb{R}^{2\times 2} \mid AM = MA \right\}.$$

i) Show that $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$ for any $M \in \mathbb{R}^{2\times 2}$.

ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4\,.$$

(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)

Solution to Exercise 2.

i) There are two ways to show this. Either one checks that $C(M)$ satisfies all the conditions in **Definition 1.3** or one uses the fact that it is the kernel of some linear map (for which we know that it is always a subspace).

First, we will use the first strategy and observe that C(M) is a subspace of $\mathbb{R}^{2\times 2}$ since

(a) We have $n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in C(M)$
Because by definition of the zero matrix,

$$nM = 0 = Mn$$

(b) We have $U+V \in C(M)$ for all $U, V \in C(M)$
As $U, V \in C(M)$, $UM = MU$ and $VM = MV$. Therefore,

$$\begin{aligned} UM + VM &= MU + MV \\ (U+V)M &= M(U+V) \end{aligned}$$

Which proves that $U + V \in C(M)$.

(c) We have $\lambda U \in C(M)$ for $\lambda \in \mathbb{R}$ and $U \in C(M)$
As $U \in C(M)$, we have $UM = MU$. Therefore,

$$\begin{aligned} \lambda \cdot UM &= \lambda \cdot MU \\ (\lambda U)M &= M(\lambda U) \end{aligned}$$

Which proves that $\lambda U \in C(M)$.

Next, we will choose the second strategy and define the map

$$\begin{aligned} F : \mathbb{R}^{2\times 2} &\longrightarrow \mathbb{R}^{2\times 2} \\ A &\longmapsto AM - MA\,. \end{aligned}$$

This map is linear, since

(a) We have $F(U+V) = F(U) + F(V)$ for all $U, V \in \mathbb{R}^{2\times 2}$:

$$\begin{aligned} F(U+V) &= (U+V)M - M(U+V) \\ &= UM + VM - MU - MV \\ &= (UM - MU) + (VM - MV) \\ &= F(U) + F(V)\,. \end{aligned}$$

(b) We have $F(\lambda \cdot U) = \lambda \cdot F(U)$ for all $U \in \mathbb{R}^{2\times 2}, \lambda \in \mathbb{R}$:

$$\begin{aligned} F(\lambda \cdot U) &= (\lambda U) \cdot M - M(\lambda U) \\ &= \lambda \cdot UM - \lambda \cdot MU \\ &= \lambda \cdot (UM - MU) \\ &= \lambda \cdot F(U)\,. \end{aligned}$$

Since $C(M) = \ker(F)$ we conclude that $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$.

(c) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ fulfill $AS = SA$. Expanding the equation gives:

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \begin{pmatrix} 0+b & -a+0 \\ 0+d & -c+0 \end{pmatrix} &= \begin{pmatrix} 0-c & 0-d \\ a+0 & b+0 \end{pmatrix} \\ \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} &= \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \\ \begin{pmatrix} c+b & -a+d \\ d-a & -(c+b) \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Therefore, $c = -b$ and $a = d$. Setting $a = \lambda$ and $c = \mu$, all possible A can be written as:

$$\begin{aligned} A &= \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \\ A &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} \\ A &= \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned}$$

Therefore, a basis of $C(S)$ is $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

(d) The only subset of $\mathbb{R}^{2\times 2}$ with dimension 0 is the set containing the zero matrix only.
However, one can observe that for all M, the equation

$$I_2 M = M I_2$$

is always true per the definition of the identity matrix.
Therefore, all scalar multiples of the identity matrix will always be elements of $C(M)$.
Another observation is that for all M, set $A = \lambda M, \lambda \in \mathbb{R}$. the equation

$$AM = (\lambda M)M = M(\lambda M) = MA$$

is always true. As such, the scalar multiples of the matrix M itself will always be elements of $C(M)$.
If M and I_2 are linearly independent, we have that

$$\dim(C(M)) \geq 2\,.$$

Otherwise if they are linearly dependent, we have $M = \lambda I_2$ for $\lambda \in \mathbb{R}$ and

$$AM = A(\lambda I_2) = \lambda A I_2 = \lambda I_2 A = MA\,.$$

Which is always true for all $A \in \mathbb{R}^{2\times 2}$ and therefore $C(M) = \mathbb{R}^{2\times 2}$, i.e. $\dim(C(M)) = 4$.
As such, $\underline{2 \leq \dim(C(M)) \leq 4 \text{ for all } M \in \mathbb{R}^{2\times 2}}$.

Exercise 3. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\mathcal{P}_3 = \{f \in \mathcal{P} \mid \deg(f) \leq 3\}\ ,$$
$$U = \{f \in \mathcal{P}_3 \mid f(-1) = 0\} \subset \mathcal{P}_3\,.$$

i) Show that U is a subspace of $\mathcal{P}_3$.

ii) Determine a basis $B = (b_1, \ldots, b_n)$ of U.

iii) Determine the coordinate vector $[f]_B$ for the function f given by $f(x) = 2(x+1)^3$.

iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)

Solution to Exercise 3.

i) If U is a subspace, it satisfies:

  i) $n \in U$.
  The neutral element of $\mathcal{P}_3$, namely the zero function (denoted by n), fulfills $n(-1) = 0$ as it maps all values of x to 0. Therefore, $n \in U$.

  ii) $\forall u, v \in U$: $u + v \in U$.
  One observes that $u + v$ is a function defined by:

$$(u+v)(x) = u(x) + v(x)\,.$$

  Evaluating this function at $x = -1$ gives:

$$(u+v)(-1) = u(-1) + v(-1) = 0 + 0 = 0\,.$$

  Which proves that $u + v \in U$.

iii) $\forall u \in U,\ \lambda \in \mathbb{R}$: $\lambda u \in U$.
Observe that λu is a function defined by:

$$(\lambda u)(x) = \lambda \cdot u(x)$$

Evaluating this function at $x = -1$ gives:

$$(\lambda u)(-1) = \lambda \cdot (0) = 0$$

Which proves that $\lambda u \in U$.

As all three criterion are fulfilled, one concludes that U is a subspace of $\mathcal{P}_\ni$.

ii) Let a generic polynomial function $f \in U$ be defined by:

$$f(x) = ax^3 + bx^2 + cx + d$$

As f(-1) = 0, one observes that

$$0 = a(-1)^3 + b(-1)^2 + c(-1) + d$$
$$0 = -a + b - c + d$$

One observes that a, b, c are free variables. Expressing d in terms of the other three, we have:

$$d = a - b + c$$

The basis of U can be found by (setting one free variable to 1, and the rest to 0 alternatingly):

Set a = 1. d = 1. $b_1 = x^3 + 1$

Set b = 1. d = -1. $b_2 = x^2 - 1$

Set c = 1. d = 1. $b_3 = x + 1$

Therefore, a basis of U can be (b_1, b_2, b_3 linearly independent because mononomials are linearly independent):

$$B = \{x^3 + 1, x^2 - 1, x + 1\}$$

iii) Let the coordinate vector $[f]_B$ be equal to $\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix}$. This means that

$$2(x+1)^3 = \lambda_1(x^3+1) + \lambda_2(x^2-1) + \lambda_3(x+1)$$
$$2x^3 + 6x^2 + 6x + 2 = \lambda_1 x^3 + \lambda_2 x^2 + \lambda_3 x + (\lambda_1 - \lambda_2 + \lambda_3)$$

The only solution to this equation is $(\lambda_1, \lambda_2, \lambda_3) = (2, 6, 6)$. Therefore we have

$$[f]_B = \begin{pmatrix}2\\ 6\\ 6\end{pmatrix}.$$

iv) One observes that the coefficients of x^3, x^2, and x can be set freely; only the coefficient of x^0 depends on the other three. As such, one can express all possible $f \in \mathcal{P}_3$ if one can 'fix' the coefficient of x^0 to the desired amount. Thus, if we set

$$\tilde{B} = B \cup \{1\} = \{x^3 + 1, x^2 - 1, x + 1, 1\}$$

and consider the span of $\tilde{B}$. There, we can freely set all four variables to any desired value. Thus,

$$\text{span}(\tilde{B}) = \text{span}\{x^3 + 1, x^2 - 1, x + 1, 1\} = \mathcal{P}_3$$