Homework 1: Vector spaces

Deadline: 2nd May (23:55 JST), 2021

Exercise 1. (4+2+2 = 8 Points) Let $V = \{x \in \mathbb{R} \mid x > 0\}$ be the set of all positive real numbers. Define on V the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$u \oplus v = uv \qquad \text{(the usual multiplication of real numbers)}$$
$$\lambda \odot v = v^\lambda \,.$$

i) Show that $(V, \oplus, \odot)$ is a vector space.
(i.e. check that the operations $\oplus$ and $\odot$ satisfy the properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$.)

ii) Determine all subspaces of $(V, \oplus, \odot)$.

iii) Find an isomorphism
$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot)\,.$$
Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

Exercise 2. (4+2+2 = 8 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \left\{A \in \mathbb{R}^{2\times 2} \mid AM = MA\right\}\,.$$

i) Show that $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$ for any $M \in \mathbb{R}^{2\times 2}$.

ii) For $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have
$$2 \leq \dim(C(M)) \leq 4\,.$$
(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)

Exercise 3. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\mathcal{P}_3 = \{f \in \mathcal{P} \mid \deg(f) \leq 3\}\,,$$
$$U = \{f \in \mathcal{P}_3 \mid f(-1) = 0\} \subset \mathcal{P}_3\,.$$

i) Show that U is a subspace of $\mathcal{P}_3$.

ii) Determine a basis $B = (b_1, \dots, b_n)$ of U.

iii) Determine the coordinate vector $[f]_B$ for the function f given by $f(x) = 2(x+1)^3$.

iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)

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