Homework 3: Determinants

Deadline: 30th May (23:55 JST), 2021

Exercise 1. (2+3+3 = 8 Points) We define the matrix

$$A = \begin{pmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ -2 & 2 & -1 \end{pmatrix}$$

and the polynomial

$$P(\lambda) = \det(A - \lambda I_3)\,,$$

where I_3 denotes the 3×3-identity matrix.

i) Calculate $\det(A)$.

ii) Find all solutions to $P(\lambda) = 0$.

iii) For each solution λ in ii) find a non-zero vector $v \in \ker(A - \lambda I_3)$ and evaluate Av. Can you observe a relationship between v, λ and A?

Solution to Exercise 1.

i) By the Laplace Expansion of the first row, we have that

$$\begin{aligned}
\det(A) &= a_{1,1} \cdot \det(A_{1,1}) - a_{1,2} \cdot \det(A_{1,2}) + a_{1,3} \cdot \det(A_{1,3}) \\
\det(A) &= 3 \cdot \det\begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix} - (-1) \cdot \det\begin{pmatrix} 2 & 0 \\ -2 & -1 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 2 & 0 \\ -2 & 2 \end{pmatrix} \\
\det(A) &= 3 \cdot 0 + 1 \cdot (-2) + 0 \\
\det(A) &= \underline{-2}
\end{aligned}$$

Note that the Laplace Expansion over any row or column of A will give the same result.

ii) First, one can write $A - \lambda I_3$ as

$$A - \lambda I_3 = \begin{pmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ -2 & 2 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3-\lambda & -1 & 0 \\ 2 & -\lambda & 0 \\ -2 & 2 & -1-\lambda \end{pmatrix}$$

The Laplace Expansion over the third column gives:

$$\begin{aligned}
\det(A - \lambda I_3) &= (-1)^{3+1} a_{1,3} \det(A_{1,3}) + (-1)^{3+2} a_{2,3} \det(A_{2,3}) + (-1)^{3+3} a_{3,3} \det(A_{3,3}) \\
\det(A - \lambda I_3) &= 0 \cdot \det(A_{1,3}) - 0 \det(A_{2,3}) + (-1-\lambda) \cdot \det\begin{pmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{pmatrix} \\
\det(A - \lambda I_3) &= (-1-\lambda) \cdot ((3-\lambda)(-\lambda) - (-1)(2)) \\
\det(A - \lambda I_3) &= (-1-\lambda) \cdot \left(-3\lambda + \lambda^2 + 2\right) \\
\det(A - \lambda I_3) &= -(\lambda+1)(\lambda-1)(\lambda-2)
\end{aligned}$$

One then concludes that the solutions to $P(\lambda) = 0$ are:

$$\lambda = \underline{-1} \quad , \quad \lambda = \underline{1} \quad \textit{and} \quad \lambda = \underline{2}\,.$$

iii) For $\lambda = -1$, let $v \in \ker(A + I_3)$. Then, v must satisfy

$$\begin{pmatrix} 4 & -1 & 0 \\ 2 & 1 & 0 \\ -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4v_1 - v_2 \\ 2v_1 + v_2 \\ -2v_1 + 2v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the three linear equations gives the result that:

$$\ker(A + I_3) = \left\{ v \in \mathbb{R}^3 \middle| v = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \quad , t \in \mathbb{R} \right\}$$

Consider the expression Av when $v = \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix}$ for an arbitrary $t \in \mathbb{R}$:

$$\begin{aligned} Av &= \begin{pmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \\ Av &= \begin{pmatrix} 0 \\ 0 \\ -t \end{pmatrix} \\ Av &= (-1) \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} \\ Av &= \underline{\lambda v} \end{aligned}$$

For $\lambda = 1$, let $v \in \ker(A - I_3)$. Then, v must satisfy

$$\begin{pmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ -2 & 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2v_1 - v_2 \\ 2v_1 - v_2 \\ -2v_1 + 2v_2 - 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the three linear equations gives the result that:

$$\ker(A - I_3) = \left\{ v \in \mathbb{R}^3 \middle| v = \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} \quad , t \in \mathbb{R} \right\}$$

Consider the expression Av when $v = \begin{pmatrix} t \\ 2t \\ t \end{pmatrix}$ for an arbitrary $t \in \mathbb{R}$:

$$\begin{aligned} Av &= \begin{pmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} \\ Av &= \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} \\ Av &= (1) \begin{pmatrix} t \\ 2t \\ t \end{pmatrix} \\ Av &= \underline{\lambda v} \end{aligned}$$

For $\lambda = 2$, let $v \in \ker(A - 2I_3)$. Then, v must satisfy

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ -2 & 2 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} v_1 - v_2 \\ 2v_1 - 2v_2 \\ -2v_1 + 2v_2 - 3v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Solving the three linear equations gives the result that:

$$\ker(A - 2I_3) = \left\{ v \in \mathbb{R}^3 \middle| v = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}, t \in \mathbb{R} \right\}$$

Consider the expression Av when $v = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$ for an arbitrary $t \in \mathbb{R}$:

$$\begin{aligned} Av &= \begin{pmatrix} 3 & -1 & 0 \\ 2 & 0 & 0 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} \\ Av &= \begin{pmatrix} 2t \\ 2t \\ 0 \end{pmatrix} \\ Av &= 2 \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} \\ Av &= \underline{\lambda v} \end{aligned}$$

Comment: As we saw in Lecture 7, what we calculated here are the eigenvalues and eigenvectors of A.

Exercise 2. (2+4 = 6 Points) (Geometric interpretation of the determinant)
We define the vectors $v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, u = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^2$.

i) Connect the endpoints of the vectors 0, v, u and $v+u$ to get a parallelogram in $\mathbb{R}^2$. (Make a sketch)

ii) Show that the area of this parallelogram is given by $\det \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$, i.e. the determinant of the matrix which has v and u as columns.

(Bonus: Show that this works in general, i.e. if you write two vectors in $\mathbb{R}^2$ into the columns of a matrix $A \in \mathbb{R}^{2\times 2}$ then $|\det(A)|$ gives the area of the parallelogram spanned by them.)

Solution to Exercise 2.

i)

$$v = \begin{pmatrix} 4 \\ 2 \end{pmatrix};\ w = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.\ \text{Thus,}\ v + w = \begin{pmatrix} 4+2 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

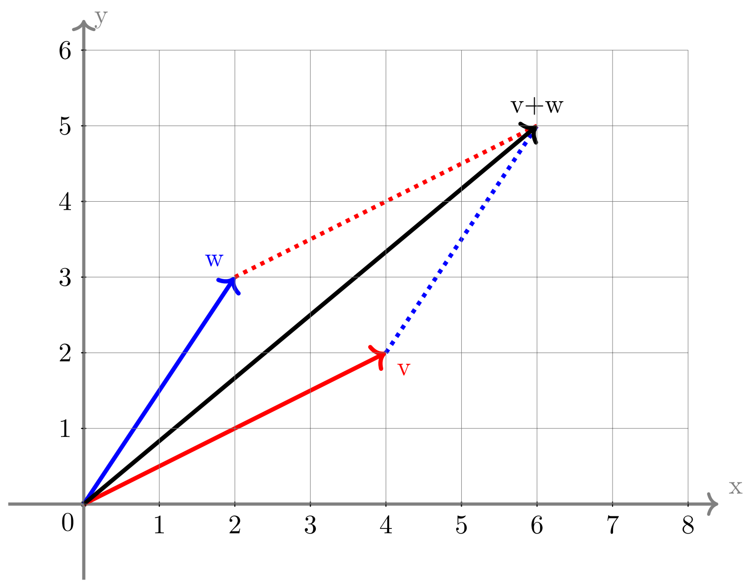


Figure 1: v, w, and v+w. The other sides of the parallelogram are the dotted lines.

ii) First, one has that

$$\det \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix} = 4(3) - 2(2) = 12 - 4 = 8$$

Next, one can find the area of the parallelogram by finding the area of the larger rectangle, then substracting the non-shaded areas (figure is on the next page).

Let A denote the area of the parallelogram itself.

$$\begin{aligned} A &= (4+2)\cdot(2+3) - 2\cdot(2\cdot 2) - 2\cdot\left(\frac{1}{2}\cdot 3\cdot 2\right) - 2\cdot\left(\frac{1}{2}\cdot 4\cdot 2\right) \\ A &= 6\cdot 5 - 2\cdot 4 - 2\cdot 3 - 2\cdot 4 \\ A &= 30 - 8 - 6 - 8 \\ A &= 8 = \underline{\det(A)} \end{aligned}$$

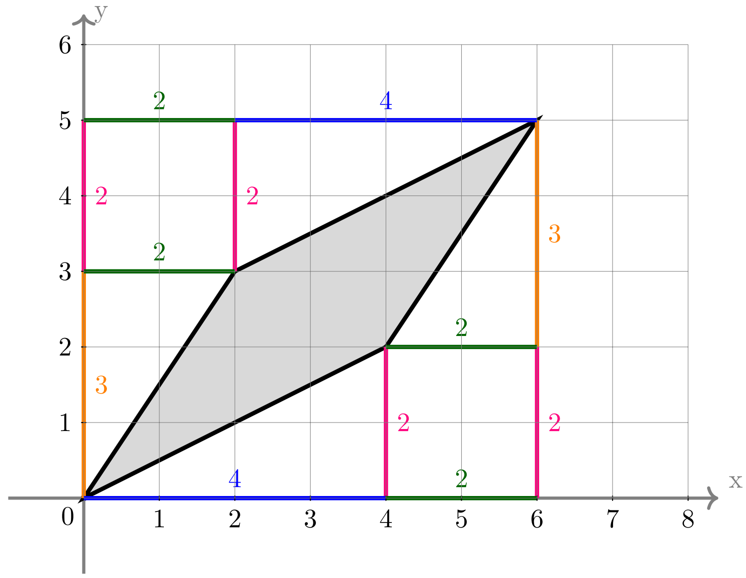


Figure 2: The parallelogram for part ii), with the lengths of the sides included

Bonus: Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$ and $w = \begin{pmatrix} c \\ d \end{pmatrix}$ be arbitrary vectors in $\mathbb{R}^2$.

For all possible vectors in $\mathbb{R}^2$, the expression $abcd$ is well definend, and can take on any real value.

Now, we divide all possible cases based on the sign of $abcd$.

Case 1: v and w are chosen such that $abcd \geq 0$. In this case, observe that

$$\begin{aligned} |ad - bc| &= \sqrt{(|ad - bc|)^2} \\ |ad - bc| &= \sqrt{(ad)^2 - 2abcd + (bc)^2} \end{aligned}$$

As $abcd \geq 0$, $abcd = |abcd|$.

$$\begin{aligned} |ad - bc| &= \sqrt{(|ad|)^2 - 2|abcd| + (|bc|)^2} \\ |ad - bc| &= \sqrt{(|ad|)^2 - 2|ab| \cdot |cd| + (|bc|)^2} \\ |ad - bc| &= \big||ad| - |bc|\big| \end{aligned}$$

Then, the area of the parallelogram (denoted as A) can be written as: (figure in next page)

$$\begin{aligned} A &= \left|(|a| + |c|)(|b| + |d|) - 2|a||d| - 2\frac{|d||c|}{2} - 2\frac{|a||b|}{2}\right| \\ A &= \Big||ab| + |bc| + |cd| + |ad| - 2|ad| - |cd| - |ad|\Big| \\ A &= \Big||bc| - |ad|\Big| \\ A &= \Big||ad| - |bc|\Big| \\ A &= \Big|ad - bc\Big| \\ A &= \underline{\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}} \end{aligned}$$

Case 2: v and w are chosen such that $abcd < 0$. In this case, observe that

$$
\begin{aligned}
|ad - bc| &= \sqrt{(|ad - bc|)^2} \\
|ad - bc| &= \sqrt{(ad)^2 - 2abcd + (bc)^2}
\end{aligned}
$$

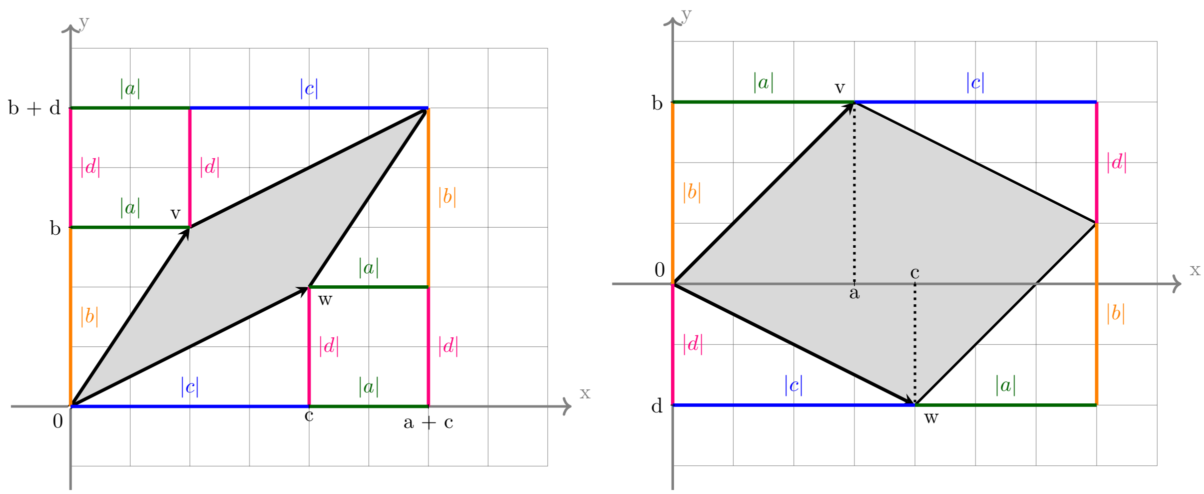
As $abcd < 0$, $abcd = -|abcd|$.

$$
\begin{aligned}
|ad - bc| &= \sqrt{(|ad|)^2 + 2|abcd| + (|bc|)^2} \\
|ad - bc| &= \sqrt{(|ad|)^2 + 2|ab| \cdot |cd| + (|bc|)^2} \\
|ad - bc| &= \Big||ad| + |bc|\Big|
\end{aligned}
$$

Then, the area of the parallelogram (denoted as A) can be written as: (figure in next page)

$$
\begin{aligned}
A &= \left|(|a| + |c|)(|b| + |d|) - 2\frac{|d||c|}{2} - 2\frac{|a||b|}{2}\right| \\
A &= \Big||ab| + |bc| + |cd| + |ad| - |cd| - |ad|\Big| \\
A &= \Big||ad| + |bc|\Big| \\
A &= \Big|ad - bc\Big| \\
A &= \underline{\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}}
\end{aligned}
$$

Which proves that the area of the parallelogram spanned by the two vectors is $\underline{\text{equal to } \left|\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right|}$.



(a) The case when $abcd \geq 0$.

(b) The case when $abcd < 0$.

Figure 3: The parallelograms for the two possible cases of the sign of $abcd$.

Exercise 3. (6+2 = 8 Points)

i) Show that the determinant is linear in each row, i.e. for any $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ and $1 \leq l \leq n$ show that the map

$$\begin{aligned} F_{A,l} : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \det(A(l;x)) \end{aligned}$$

is linear. Here $A(l;x)$ denotes the matrix A, where the l-th row is replaced by the vector x^T. (See at the bottom of page 7 in the overview notes)

ii) Assume that A is invertible. What is the kernel of $F_{A,n}$?

Solution to Exercise 3.

i) Let $B := A(l : x)$. We observe that by definition, $b_{l,\sigma(l)} = x_{\sigma(l)}$ for all $\sigma \in S_n$.
One can write the determinant of B as

$$\det(B) = \sum_{\sigma\in\S_n} \text{sign}(\sigma) \prod_{\substack{j=1\\ j\neq l}}^{l-1} b_{j,\sigma(j)} \prod_{j=l}^{l} b_{j,\sigma(j)}$$

$$\det(B) = \sum_{\sigma\in\S_n} \left(\text{sign}(\sigma) \prod_{\substack{j=1\\ j\neq l}}^{l-1} b_{j,\sigma(j)} \right) x_{\sigma(j)}$$

One observes that all the terms other x_i is independent of x_i.

One can then now prove that the $F_{A:l}$ is a linear map:

i) $F_{A:l}(x+y) = F_{A:l}(x) + F_{A:l}(y)$. By the expression obtained for $\det(A(l : x)) = \det(B)$, we have

$$\begin{aligned} F_{A:l}(x+y) &= \sum_{\sigma\in\S_n} \left(\text{sign}(\sigma) \prod_{\substack{j=1\\ j\neq l}}^{l-1} b_{j,\sigma(j)} \right) (x_{\sigma(j)} + y_{\sigma(j)}) \\ F_{A:l}(x+y) &= \sum_{\sigma\in\S_n} \left(\text{sign}(\sigma) \prod_{\substack{j=1\\ j\neq l}}^{l-1} b_{j,\sigma(j)} \right) x_{\sigma(j)} + \sum_{\sigma\in\S_n} \left(\text{sign}(\sigma) \prod_{\substack{j=1\\ j\neq l}}^{l-1} b_{j,\sigma(j)} \right) y_{\sigma(j)} \\ F_{A:l}(x+y) &= F_{A:l}(x) + F_{A:l}(y) \end{aligned}$$

ii) $F_{A:l}(\mu x) = \mu F_{A:l}(x)$, for $\mu \in \mathbb{R}$. By the expression obtained for $\det(A(l : x)) = \det(B)$, we have

$$\begin{aligned} F_{A:l}(\mu x) &= \sum_{\sigma\in\S_n} \left(\text{sign}(\sigma) \prod_{\substack{j=1\\ j\neq l}}^{l-1} b_{j,\sigma(j)} \right) \mu x_{\sigma(j)} \\ F_{A:l}(\mu x) &= \mu \cdot \sum_{\sigma\in\S_n} \left(\text{sign}(\sigma) \prod_{\substack{j=1\\ j\neq l}}^{l-1} b_{j,\sigma(j)} \right) x_{\sigma(j)} \\ F_{A:l}(\mu x) &= \mu F_{A:l}(x) \end{aligned}$$

Which proves that $F_{A:l}$ is a linear map.

ii) Let us rewrite the matrix A as:

$$A = \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ & \vdots & \\ - & r_n & - \end{pmatrix}$$

We have that if two rows of the same matrix A are identical, we have $\det(A) = 0$.

As such, one can infer that $r_1 \neq r_2 \neq \ldots \neq r_n$.

Therefore, if we set $x^T = \{-r_i-\}$ for any $1 \leq i \leq n-1$, we get that

$$A(n:x) = \begin{pmatrix} - & r_1 & - \\ - & r_2 & - \\ & \vdots & \\ - & r_i & - \end{pmatrix}$$

As here, one of the matrices' rows are identical, $\det(A(n:x)) = 0$.

One also observes that if x is the zero matrix, then $\det(A(n:x)) = 0$ as it has a zero (or empty) row.

One also observes that for x that is any linear combination of $-r_1-$ to $-r_{n-1}-$ will also be in the kernel of this linear map.

$$\ker(F_{A:n}) = \underline{\text{span}\{\begin{pmatrix} - & r_1 & - \end{pmatrix}^T, \ldots, \begin{pmatrix} - & r_{n-1} & - \end{pmatrix}^T\}}$$