Final exam

Good luck Viel Erfolg 好好儿考啊 시험 잘봐 Semoga sukses амжилт хусье
頑張って Chúc may mắn nhé Bugünkü sınavında iyi şanslar. Semoga berjaya Galingan nyo

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Exercise 1. (14 Points) We define the following matrix

$$A = \begin{pmatrix} -4 & -2 & 5 \\ 3 & 1 & 5 \\ 0 & 0 & -2 \end{pmatrix}.$$

i) Compute the determinant of A.

ii) Find all eigenvalues of A. For each eigenvalue λ determine a basis of the eigenspace $E_\lambda(A)$.

iii) Is A diagonalizable and/or invertible? Justify your answers.

Exercise 2. (12 Points) Decide if the following statements are true or false. Justify your answer by giving a short explanation. In iii) V is a finitely generated vector space.

i) The set $U = \{p \in \mathcal{P}_2 \mid p(1) \geq p(2)\}$ is a subspace of $\mathcal{P}_2$.

ii) The set $U = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \mid a + d = 0 \right\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

iii) For any subspace $U \subset V$, there exists a linear map $F : V \to V$ with $\mathrm{im}(F) = U$.

iv) If $A \in \mathbb{R}^{n\times n}$ has n different eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ then $\det(A) = \lambda_1 \cdots \lambda_n$.

v) For any matrix $A \in \mathbb{R}^{n\times n}$ the matrix $A + A^T$ is diagonalizable.

vi) Let T be a differential operator of order n and $g \in \mathcal{P}_n$. Then the equation $T(f) = g$ always has at least one solution $f \in \mathcal{P}_n$.

Exercise 3. (8 Points) Give an example of

i) an orthogonal matrix $B \in \mathbb{R}^{2\times 2}$ which is not diagonalizable.

ii) a linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ with $\ker(F) = \mathrm{im}(F)$.

iii) a linear map $H : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ without any eigenvalues.

Justify your examples.

Exercise 4. (8 Points) Let $x_n \in \mathbb{R}^2$ for $n \geq 0$ be defined by

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{n+1} = M x_n\,, \quad \text{where} \quad M = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}.$$

Determine an explicit formula for x_n.

Exercise 5. (8 Points) Find all solutions to the following differential equation

$$f'''(t) - 4f''(t) + 4f'(t) = 8t\,,$$

which satisfy $f(0) = 2021$.

After finishing this exam, submit it to NUCT (Assignment: Final Exam) before 27th July 12:20 JST.