Math. Tutorial 2b, Linear Algebra II
30th April 2020

i) $v_1, \dots, v_n$ *are linearly independent.*

ii) $V = \operatorname{span}\{v_1, \dots, v_n\}$.

iii) $(v_1, \dots, v_n)$ *is a basis of* V.

Proposition 1.11. *Let* V *be finitely generated and* $U \subset V$ *a subspace. Then* U *is also finitely generated.*

Proposition 1.12. *Let* $B = (v_1, \dots, v_n)$ *be a basis of* V. *Then for all* $u \in V$ *there exist unique* $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, *such that*

$$u = \sum_{i=1}^{n} \lambda_i v_i \,.$$

Definition 1.13. *Let* $B = (v_1, \dots, v_n)$ *be a basis of* V.

i) The $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ *in Proposition 1.12 are called the* **coordinates** *of* $u \in V$ *in the basis* B.

ii) The vector $[u]_B \in \mathbb{R}^n$ *given by*

$$[u]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is called the **coordinate vector** *of* u *with respect to the basis* B.

2 Linear maps

Definition 2.1. *Let* V, W *be vector spaces. A* **linear map** *is a function* $F : V \to W$ *satisfying*

i) $F(u + v) = F(u) + F(v)$ *for all* $u, v \in V$.

ii) $F(\lambda \cdot u) = \lambda \cdot F(u)$ *for all* $u \in V, \lambda \in \mathbb{R}$.

Definition 2.2. *Let* $F : V \to W$ *be a linear map.*

i) The **kernel of** F *is given by*

$$\ker(F) = \{u \in V \mid F(u) = 0\} \subset V \,.$$

ii) The **image of** F *is given by*

$$\operatorname{im}(F) = \{w \in V \mid \exists u \in V : w = F(u)\} \subset W \,.$$

With the same arguments as in the $\mathbb{R}^n$*-case we see that* $\ker(F)$ *is a subspace of* V *and* $\operatorname{im}(F)$ *is a subspace of* W. *If* $\operatorname{im}(F)$ *is finitely generated, we define the* **rank of** F *by* $\operatorname{rk}(F) = \dim(\operatorname{im}(F))$.

Theorem 2.3 (kernel-image theorem)**.** *Let V be finitely generated and let $F : V \to W$ be a linear map to an arbitrary vector space W. Then*

$$\dim V = \dim(\ker(F)) + \dim(\mathrm{im}(F)) .$$

Definition 2.4. *i) (Recall) A function $f : X \to Y$ is* **invertible** *if there exist a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. f is invertible iff f is bijective, i.e. injective and surjective.*

ii) An invertible linear map $F : V \to W$ is called an **isomorphism**.

iii) Two vector spaces V and W are called **isomorphic** *(Notation: $V \cong W$) if there exists an isomorphism $F : V \to W$.*

Theorem 2.5. *i) A linear map $F : V \to W$ is an isomorphism iff $\ker(F) = \{0\}$ (F is injective) and $\mathrm{im}(F) = W$ (F is surjective).*

ii) Let $F : V \to W$ be an isomorphism and $(b_1, \ldots, b_n)$ a basis of V. Then $(F(b_1), \ldots, F(b_n))$ is a basis of W.

iii) Let V, W be finitely generated and $V \cong W$ then $\dim(V) = \dim(W)$.

iv) Let V, W be finitely generated and $\dim(V) = \dim(W)$. Then for a linear map $F : V \to W$ the following three statements are equivalent

(a) F is an isomorphism.

(b) $\ker(F) = \{0\}$.

(c) $\mathrm{im}(F) = W$.

Proposition 2.6. *Let V be finitely generated with basis $B = (b_1, \ldots, b_n)$, i.e. $\dim(V) = n$. Then the* **coordinate map**

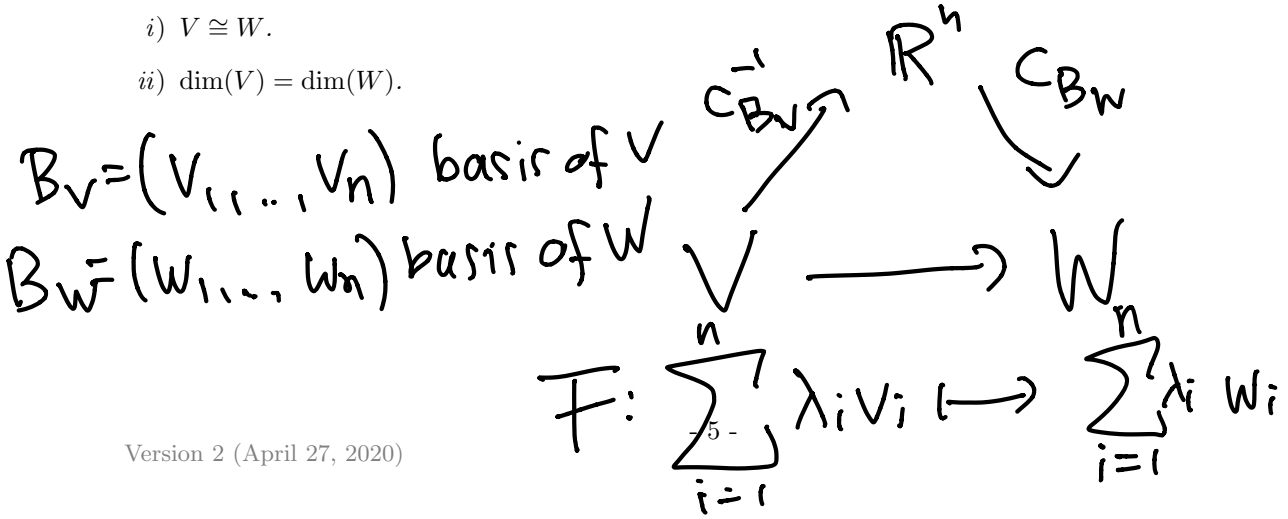
$$c_B : \mathbb{R}^n \longrightarrow V ,$$
$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \longmapsto \sum_{i=1}^{n} \lambda_i b_i$$

is an isomorphism. The inverse is given by $c_B^{-1}(u) = [u]_B$ for $u \in V$.

Corollary 2.7. *Let V, W be finitely generated. Then the following two statements are equivalent*

i) $V \cong W$.

ii) $\dim(V) = \dim(W)$.



Homework 2: Linear maps

Deadline: 17th May, 2020

Recall: $\mathcal{P}_n$ is the vector space of polynomial functions of degree $\leq n$.

Exercise 1. (12 Points) For $n \geq 1$ we define the map $F_n : \mathcal{P}_n \to \mathbb{R}^3$ for a $p \in \mathcal{P}_n$ by

$$F_n(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix} .$$

i) Show that F_n is a linear map.

ii) Show that F_2 is invertible, i.e. an isomorphism.

iii) Calculate the inverse of F_2.

iv) Check if F_1 and F_3 are injective and/or surjective.

v) Determine a basis of $\mathrm{im}(F_1)$.

Exercise 2. (8 Points) We define the map $G : \mathbb{R}^3 \to \mathcal{P}_2$ by

$$G \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + (1-b)x^2 + cx^2 .$$

i) Show that G is a linear map.

ii) Calculate a basis for $\ker(G)$ and $\mathrm{im}(G)$ and determine their dimensions.

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Example for Corollary 2.7

$V = P_2 \qquad B_V = (\underbrace{x^2+1}_{v_1}, \underbrace{2x+1}_{v_2}, \underbrace{1}_{v_3})$

$W = \mathbb{R}^3 \qquad B_W = (\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{w_2}, \underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}}_{w_3})$

$T: V \longrightarrow W$

$p(x) = 3x^2+5 \qquad T(p) = 3 \cdot w_1 + 0 \cdot w_2 + 2 \cdot w_3$

$p = 3v_1 + 0v_2 + 2v_3 \qquad = \begin{pmatrix}3\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\end{pmatrix} + \begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}5\\2\\2\end{pmatrix}$

Exercise 1. (12 Points) For $n \geq 1$ we define the map $F_n : \mathcal{P}_n \to \mathbb{R}^3$ for a $p \in \mathcal{P}_n$ by

$$F_n(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix}.$$

i) Show that F_n is a linear map.

ii) Show that F_2 is invertible, i.e. an isomorphism.

iii) Calculate the inverse of F_2.

iv) Check if F_1 and F_3 are injective and/or surjective.

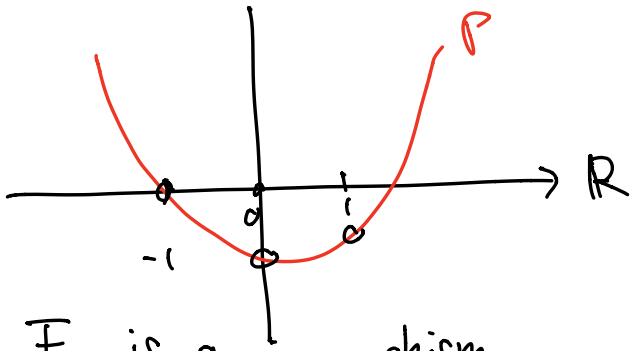
v) Determine a basis of $\mathrm{im}(F_1)$.

$n=2 \quad p \in P_2$

$p(x) = 2x^2 + x - 4$

$F_2(p) = \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix}$

i) $g, p \in P_n \quad F_n(g+p) = F_n(g) + F_n(p)$



ii) Want to show F_2 is an isomorphism

$\Leftrightarrow F_2$ is inj. & surj.

$\ker F_2 = \{0\}$

F_2 is inj: If $F_2(p) = 0 \Rightarrow p = 0$

$a = b = c = 0$

$p \in P_2$

$p(x) = ax^2 + bx + c$

$$F_2(p) = \begin{pmatrix} p(-1) \\ p(0) \\ p(1) \end{pmatrix} = \begin{pmatrix} a - b + c \\ c \\ a + b + c \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Exercise 2. (8 Points) We define the map $G : \mathbb{R}^3 \to \mathcal{P}_2$ by

$$G\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a + (1-b)x^2 + cx^2 \,.$$

$= __ x^2 + __ x + __$

i) Show that G is a linear map.

ii) Calculate a basis for $\ker(G)$ and $\mathrm{im}(G)$ and determine their dimensions.

i) $G(u+v) = G\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) \overset{?}{=} G\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + G\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$G(\lambda \cdot v) = \lambda \cdot G(v)$ $=$

$G\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} =$

ii) $\ker(G) = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; G(v) = 0 \right\}$ v $=$ $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$