# Review for the final exam

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The following exercises are for the preparation of the final exam. The final exam will contain less exercises! Also good for the preparation: Homework exercises 1.3, 3.2, 4.2, 5.2, 6.1, 6.2, 7.2, 7.3, 8.1, 9.3.

**Exercise 1.** We define the following matrix

$$A = \begin{pmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 3 & 2 \end{pmatrix}.$$

i) Compute the determinant of $A$.

ii) Find all eigenvalues of $A$. For each eigenvalue $\lambda$ determine a basis of the eigenspace $E_\lambda(A)$.

iii) Is $A$ diagonalizable and/or invertible and/or orthogonal ? Justify your answers.

**Exercise 2.** Decide if the following statements are true or false. Justify your answer by giving a short explanation (e.g. give a a counterexample in the case when it is false). $V$ is always a finitely generated vector space and $A, B \in \mathbb{R}^{n\times n}$ are matrices.

1) The set $U = \{p \in \mathcal{P}_2 \mid p(1) = p(2)\}$ is a subspace of $\mathcal{P}_2$.

2) The set $U = \left\{A \in \mathbb{R}^{2\times 2} \mid A \text{ has eigenvalue } 1\right\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

3) Every $3 \times 3$ matrix has at least one eigenvalue.

4) For any subspace $U \subset V$, there exist a linear map $F : V \to V$ with $\ker(F) = U$.

5) There exists a linear map $F : \mathbb{R}^3 \to \mathbb{R}^3$ with $\dim(\ker(F)) = \dim(\mathrm{im}(F)) = 2$.

6) If $(b_1, b_2)$ and $(c_1, c_2)$ are two bases of a vector space $V$, then $(b_1 + c_1, b_2 - c_2)$ is also a basis of $V$.

7) If $\det(A) = 0$, then $A$ has at least one eigenvalue.

8) If $\det(A) = 0$ and $\det(B) = 0$ then $\det(A + B) = 0$.

9) There exist symmetric matrices without eigenvalues.

10) Every orthogonal matrix is invertible.

11) Let $F : V \to V$ be a linear map. If $F \circ F$ has at least one eigenvalue, then $F$ also has at least one eigenvalue.

**Exercise 3.** Give an example of

i) a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$ with $\dim(\ker(F)) = \dim(\mathrm{im}(F)) = 1$.

ii) a matrix $A$ which is not symmetric but diagonalizable.

iii) a linear map $F : \mathcal{P}_1 \to \mathcal{P}_1$ without eigenvalues.

iv) an orthogonal matrix $A \in \mathbb{R}^{2\times 2}$ without 0 as an entry.


Version: July 27, 2020

**Exercise 4.** Let $x_n \in \mathbb{R}^2$ be for $n \geq 0$ be defined by

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad x_{n+1} = Mx_n\,, \qquad \text{where} \qquad M = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

Determine an explicit formula for $x_n$.

**Exercise 5.** Let $F : \mathcal{P}_2 \to \mathcal{P}_2$ be the linear map given by

$$F(p)(t) = tp'(t) + 2p(t)\,.$$

Calculate the determinant of $F$.

**Exercise 6.** Let $V$ be a finitely generated vector space and $P : V \to V$ be a linear map with $P \circ P = P$. Assume that $\lambda$ is an eigenvalue of $P$. What are the possible values of $\lambda$?

**Exercise 7.** Find all solutions to the following differential equation

$$f''' - 2f'' = 4\,,$$

such that $f(0) = f'(0) = f''(0) = 0$.