Linear Algebra II

Overview notes

G30 Program, Nagoya University (Spring 2020)

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Lecture notes and exercises are available at: `https://www.henrikbachmann.com/la2_2020.html`

These notes serve as a compact overview of the definitions, propositions, lemmas, corollaries, and theorems given in the lectures. It will be updated regularly (This is Version 13 from July 21, 2020). The proofs, examples, and explanations are provided in the handwritten notes/lectures. The reference book for this course is [B], and we will probably cover Chapters 4,6,7 and 9 during this semester.

If you find any typos in this note, please let me know!

Contents

References

[B] O. Bretscher: *Linear Algebra with Applications*, 4th edition, Pearson 2009.

1 Vector spaces

Definition 1.1. *A* **(real) vector space** *(linear space) is a set V together with two functions*

Addition	*Scalar multiplication*
$\begin{aligned} + : V \times V &\longrightarrow V \\ (u, v) &\longmapsto u + v \end{aligned}$	$\begin{aligned} \cdot : \mathbb{R} \times V &\longrightarrow V \\ (\lambda, v) &\longmapsto \lambda \cdot v = \lambda v \end{aligned}$

satisfying the following properties:

- *Properties of the addition:*

(A.1) $\forall u, v, w \in V$*:* $(u + v) + w = u + (v + w)$. *(Associativity)*

(A.2) $\forall u, v \in V$*:* $u + v = v + u$. *(Commutativity)*

(A.3) $\exists n \in V, \forall u \in V$*:* $n + u = u$. *(Identity/neutral element of addition)*

(A.4) $\forall u \in V, \exists v \in V$*:* $u + v = n$. *(Inverse elements of addition)*

- *Compatibility of addition and scalar multiplication:*

(C.1) $\forall u, v \in V, \lambda \in \mathbb{R}$*:* $\lambda(u + v) = \lambda u + \lambda v$. *(Distributivity I)*

(C.2) $\forall u \in V, \lambda, \mu \in \mathbb{R}$*:* $(\lambda + \mu)u = \lambda u + \mu u$. *(Distributivity II)*

(C.3) $\forall u \in V, \lambda, \mu \in \mathbb{R}$*:* $\lambda(\mu u) = (\lambda\mu)u$.

(C.4) $\forall u \in V$*:* $1u = u$.

Proposition 1.2. *Let V be a vector space and $u \in V$.*

i) $u + n = u$.

ii) *If n, $\tilde{n} \in V$ both satisfy (A.3) in Definition 1.1, then $n = \tilde{n}$.*
(The Identity element is unique)

iii) *If for a fixed $u \in V$ the elements $v, \tilde{v} \in V$ both satisfy (A.4), i.e. $u + v = u + \tilde{v} = n$, then $v = \tilde{v}$.*
(The inverse of an element u is unique)

iv) $u + (-1)u = 0$.

The identity (also called neutral) element $n \in V$ of a vector space is usually (by abuse of notation) also denoted by 0. Be always aware in the following if 0 means the real number 0 or the identity element of a vector space. (These are two different things!)

Definition 1.3. *Let V be a vector space. A subset $U \subset V$ is a* **subspace** *if*

i) $0 \in U$.

ii) $\forall u, v \in U$*:* $u + v \in U$.

iii) $\forall u \in V, \lambda \in \mathbb{R}$*:* $\lambda u \in U$.

Proposition 1.4. *If $U \subset V$ is a subspace, then U is also a vector space with the operations inherited from V.*

Definition 1.5. *Let V be a vector space and $v_1, \dots, v_n \in V$.*

i) *The* **span** *of the elements $v_1, \dots, v_n$ is given by the set of all their* **linear combinations**, *i.e.*

$$\operatorname{span}\{v_1, \dots, v_n\} = \left\{ \sum_{i=1}^{n} \lambda_i v_i \in V \;\middle|\; \lambda_1, \dots, \lambda_n \in \mathbb{R} \right\}.$$

ii) *The elements $v_1, \dots, v_n$* **span (or generate) the space** *V if $\operatorname{span}\{v_1, \dots, v_n\} = V$.*

iii) *V is* **finitely generated** *if there exist $v_1, \dots, v_n \in V$ with $\operatorname{span}\{v_1, \dots, v_n\} = V$.*
(i.e. one just needs finitely many elements to generate the space)

iv) *The elements $v_1, \dots, v_n$ are* **linearly independent** *if*

$$\lambda_1 v_1 + \dots + \lambda_n v_n = 0 \implies \lambda_1 = \ldots = \lambda_n = 0.$$

v) *$B = (v_1, \dots, v_n)$ is a* **basis** *of V if $v_1, \dots, v_n$ are linearly independent and $\operatorname{span}\{v_1, \dots, v_n\} = V$.*

Proposition 1.6. *Let V be a vector space and $v_1, \dots, v_n \in V$. The following statements are equivalent.*

i) *$v_1, \dots, v_n$ are linearly dependent.*

ii) *There exist a $1 \le j \le n$ such that $v_j \in \operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\}$.*

iii) *There exist a $1 \le j \le n$ such that $\operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\} = \operatorname{span}\{v_1, \dots, v_n\}$.*

Lemma 1.7. *If $v_1, \dots, v_l \in V$ are linearly independent and $V = \operatorname{span}\{w_1, \dots, w_m\}$, then $l \le m$.*

Theorem 1.8. *Let V be a finitely generated vector space. Then we have the following*

i) *V has a (finite) basis.*

ii) *All bases of V have the same number of elements.*

iii) *If $v_1, \dots, v_l \in V$ are linearly independent then there exist $v_{l+1}, \dots, v_n \in V$ such that $(v_1, \dots, v_n)$ is a basis of V.*

iv) *If $V = \operatorname{span}\{w_1, \dots, w_m\}$, then there exist a subset $\{u_1, \dots, u_l\} \subset \{w_1, \dots, w_m\}$, such that $(u_1, \dots, u_l)$ is a basis of V.*

Definition 1.9. *Let V be a finitely generated vector space with basis $(v_1, \dots, v_n)$. Then $\dim(V) = n$ is the* **dimension of V**.

Corollary 1.10. *Let V be a vector space with $\dim(V) = n$ and $v_1, \dots, v_n \in V$. Then the following statements are equivalent.*

i) $v_1, \ldots, v_n$ *are linearly independent.*

ii) $V = \mathrm{span}\{v_1, \ldots, v_n\}$.

iii) $(v_1, \ldots, v_n)$ *is a basis of* V.

Proposition 1.11. *Let* V *be finitely generated and* $U \subset V$ *a subspace. Then* U *is also finitely generated.*

Proposition 1.12. *Let* $B = (v_1, \ldots, v_n)$ *be a basis of* V. *Then for all* $u \in V$ *there exist unique* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, *such that*

$$u = \sum_{i=1}^{n} \lambda_i v_i \,.$$

Definition 1.13. *Let* $B = (v_1, \ldots, v_n)$ *be a basis of* V.

i) *The* $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ *in Proposition 1.12 are called the* **coordinates** *of* $u \in V$ *in the basis* B.

ii) *The vector* $[u]_B \in \mathbb{R}^n$ *given by*

$$[u]_B = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is called the **coordinate vector** *of* u *with respect to the basis* B.

2 Linear maps

Definition 2.1. *Let* V, W *be vector spaces. A* **linear map** *is a function* $F : V \to W$ *satisfying*

i) $F(u + v) = F(u) + F(v)$ *for all* $u, v \in V$.

ii) $F(\lambda \cdot u) = \lambda \cdot F(u)$ *for all* $u \in V, \lambda \in \mathbb{R}$.

Definition 2.2. *Let* $F : V \to W$ *be a linear map.*

i) *The* **kernel of** F *is given by*

$$\ker(F) = \{u \in V \mid F(u) = 0\} \subset V \,.$$

ii) *The* **image of** F *is given by*

$$\mathrm{im}(F) = \{w \in V \mid \exists u \in V : w = F(u)\} \subset W \,.$$

With the same arguments as in the $\mathbb{R}^n$*-case we see that* $\ker(F)$ *is a subspace of* V *and* $\mathrm{im}(F)$ *is a subspace of* W. *If* $\mathrm{im}(F)$ *is finitely generated, we define the* **rank of** F *by* $\mathrm{rk}(F) = \dim(\mathrm{im}(F))$.

Theorem 2.3 (kernel-image theorem)**.** *Let V be finitely generated and let $F : V \to W$ be a linear map to an arbitrary vector space W. Then*

$$\dim V = \dim(\ker(F)) + \dim(\operatorname{im}(F))\,.$$

Definition 2.4. *i) (Recall) A function $f : X \to Y$ is* **invertible** *if there exist a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. f is invertible iff f is bijective, i.e. injective and surjective.*

ii) An invertible linear map $F : V \to W$ is called an **isomorphism***.*

iii) Two vector spaces V and W are called **isomorphic** *(Notation: $V \cong W$) if there exists an isomorphism $F : V \to W$.*

Theorem 2.5. *i) A linear map $F : V \to W$ is an isomorphism iff $\ker(F) = \{0\}$ (F is injective) and $\operatorname{im}(F) = W$ (F is surjective).*

ii) Let $F : V \to W$ be an isomorphism and $(b_1, \dots, b_n)$ a basis of V. Then $(F(b_1), \dots, F(b_n))$ is a basis of W.

iii) Let V, W be finitely generated and $V \cong W$ then $\dim(V) = \dim(W)$.

iv) Let V, W be finitely generated and $\dim(V) = \dim(W)$. Then for a linear map $F : V \to W$ the following three statements are equivalent

(a) F is an isomorphism.

(b) $\ker(F) = \{0\}$.

(c) $\operatorname{im}(F) = W$.

Proposition 2.6. *Let V be finitely generated with basis $B = (b_1, \dots, b_n)$, i.e. $\dim(V) = n$. Then the* **coordinate map**

$$c_B : \mathbb{R}^n \longrightarrow V\,,$$
$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \longmapsto \sum_{i=1}^{n} \lambda_i b_i$$

is an isomorphism. The inverse is given by $c_B^{-1}(u) = [u]_B$ for $u \in V$.

Corollary 2.7. *Let V, W be finitely generated. Then the following two statements are equivalent*

i) $V \cong W$.

ii) $\dim(V) = \dim(W)$.

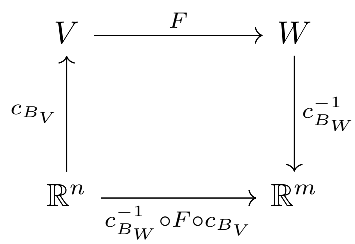
3 The matrix of a linear map

In the following V and W are finitely generated vector spaces.

Definition 3.1. *Let $B_V = (v_1, \ldots, v_n)$ be a basis of V, $B_W = (w_1, \ldots, w_m)$ be a basis of W and let $F : V \to W$ be a linear map. The* **matrix of F with respect to B_V and B_W** *is defined by*

$$[F]_{B_V}^{B_W} = \left[c_{B_W}^{-1} \circ F \circ c_{B_V}\right].$$

Here $c_{B_W}^{-1} \circ F \circ c_{B_V}$ is the linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ for which the corresponding matrix was defined before. We have the following diagram



We have

$$[F]_{B_V}^{B_W} = \begin{pmatrix} | & \cdots & | \\ [F(v_1)]_{B_W} & \cdots & [F(v_n)]_{B_W} \\ | & \cdots & | \end{pmatrix}.$$

In other words: The j-th column of $[F]_{B_V}^{B_W}$ is given by the vector $\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$, where $F(v_j) = \sum_{i=1}^m \lambda_i w_i$.

Definition 3.2. *Let $B_1 = (v_1, \ldots, v_n)$ and $B_2 = (u_1, \ldots, u_n)$ be bases of V. The* **change-of-basis matrix from B_1 to B_2** *is the matrix*

$$S_{B_1}^{B_2} = [\mathrm{id}_V]_{B_1}^{B_2} = [c_{B_2}^{-1} \circ c_{B_1}] = \begin{pmatrix} | & \cdots & | \\ [v_1]_{B_2} & \cdots & [v_n]_{B_2} \\ | & \cdots & | \end{pmatrix}.$$

4 Determinants

Definition 4.1. *A* **pattern** *in an $n \times n$-matrix is a choice of entries, such that precisely one entry from each row and each column is chosen.*

Definition 4.2. *i) A bijective map $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is called a* **permutation** *of $\{1, \dots, n\}$.*

ii) S_n denotes the set of all permutations of $\{1, \dots, n\}$.

Patterns in an $n \times n$-matrix corresponds exactly to the permutations of $\{1, \dots, n\}$. For each $\sigma \in S_n$ we have the pattern

$$P = \left\{(1, \sigma(1)), (2, \sigma(2)), \dots, (1, \sigma(n))\right\},$$

where (i, j) denotes the choice of the i-th row and the j-th column.

Definition 4.3. *i) The* **number of inversion** *of a permutation $\sigma \in S_n$, denoted by $\operatorname{inv}(\sigma)$, is the number of pairs $(i, \sigma(i)), (j, \sigma(j))$ with $i < j$ and $\sigma(i) > \sigma(j)$.*

ii) The sign of a permutation $\sigma \in S_n$ is defined by

$$\operatorname{sign}(\sigma) = (-1)^{\operatorname{inv}(\sigma)}.$$

Definition 4.4. *The* **determinant** *of a $n \times n$-matrix $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ is defined by*

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

4.1 Properties of determinants

Lemma 4.5. *For all $\sigma \in S_n$ we have $\operatorname{inv}(\sigma) = \operatorname{inv}(\sigma^{-1})$.*

Proposition 4.6. *For any $A \in \mathbb{R}^{n\times n}$ we have $\det(A) = \det(A^T)$.*

For $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ define for a vector $x \in \mathbb{R}^n$ and $1 \le l \le n$ the matrix $A(l; x)$ as the matrix where the l-th row of A gets replaced by x, i.e.

$$A(l;x) = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & & & \vdots \\ a_{l-1,1} & a_{l-1,2} & \cdots & a_{l-1,n} \\ x_1 & x_2 & \cdots & x_n \\ a_{l+1,1} & a_{l+1,2} & \cdots & a_{l+1,n} \\ \vdots & & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \in \mathbb{R}^{n\times n}, \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Proposition 4.7. *For any* $A = (a_{i,j}) \in \mathbb{R}^{n\times n}$ *and* $1 \le l \le n$ *the map*

$$\begin{aligned} F_{A,l} : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ x &\longmapsto \det(A(l;x)) \end{aligned}$$

is a linear map, i.e. the determinant is linear in each row,

Proposition 4.8. *For* $A \in \mathbb{R}^{n\times n}$ *let* $B \in \mathbb{R}^{n\times n}$ *be a matrix obtained from the matrix* A *by swapping two rows. Then we have*

$$\det(A) = -\det(B)\,.$$

Corollary 4.9. *If a matrix* $A \in \mathbb{R}^{n\times n}$ *contains two equal rows or columns, then* $\det(A) = 0$.

Recall from Linear Algebra I that there are three types of **row operations** for a matrix $A \in \mathbb{R}^{n\times n}$. $(1 \le i, j \le n\,, i \ne j\,, \lambda \in \mathbb{R})$.

(R1) Add λ-times the j-th row to the i-th row.

(R2) For $\lambda \ne 0$ multiply the i-th row with λ.

(R3) Swap the j-th row with the i-th row.

Two matrices $A, B \in \mathbb{R}^{n\times n}$ are called **row equivalent**, if one can obtain B from A by using the row operations (R1), (R2) and (R3). Notation: $A \sim B$.

Proposition 4.10. *Let* $A, B \in \mathbb{R}^{n\times n}$.

i) If B *is obtained from* A *by using (R1), then* $\det(B) = \det(A)$.

ii) If B *is obtained from* A *by using (R2), then* $\det(B) = \lambda \det(A)$.

iii) If B *is obtained from* A *by using (R3), then* $\det(B) = -\det(A)$.

Theorem 4.11. *A matrix* $A \in \mathbb{R}^{n\times n}$ *is invertible if and only if* $\det(A) \ne 0$.

Theorem 4.12. *i) For all* $A, B \in \mathbb{R}^{n\times n}$ *we have* $\det(AB) = \det(A)\det(B)$.

ii) If A *is invertible, then* $\det(A^{-1}) = \frac{1}{\det(A)}$.

Corollary 4.13. *Let* V *be a finitely generated vector space,* $F : V \to V$ *a linear map and* B_1, B_2 *two bases of* V*. Then*

$$\det\left([F]_{B_1}\right) = \det\left([F]_{B_2}\right)\,,$$

where $[F]_B = [F]_B^B$ *denotes the matrix of* F *with respect to the basis* B *(Definition 3.1).*

Definition 4.14. *Let* V *be a finitely generated vector space,* $F : V \to V$ *a linear map and* B *any basis of* V*. We define the* **determinant of the linear map** F *by*

$$\det(F) = \det\left([F]_B\right)\,.$$

Definition 4.15. *For $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ and $1 \leq i, j \leq n$ we define the* **elementary matrices** $R_i^{\lambda,j}, R_i^\lambda, R_{i,j} \in \mathbb{R}^{n\times n}$ *by*

$$R_i^{\lambda,j} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & \lambda & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, \quad R_i^\lambda = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda & \\ & & & & 1 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}, \quad R_{i,j} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & & 1 \\ & & & \ddots & \\ & & 1 & & 0 \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix}.$$

Here the λ in $R_i^{\lambda,j}$ is in the i-th row and j-th column, in R_i^λ it is in the i-th row, and in $R_{i,j}$ the 0 are on the diagonal in the i-th row and j-th column.

Proposition 4.16. *Let $A \in \mathbb{R}^{n\times n}$.*

i) $R_i^{\lambda,j}A$ is the matrix obtained from A by row operation (R1). *(Add λ-times the j-th row to the i-th row)*

ii) $R_i^\lambda A$ is the matrix obtained from A by row operation (R2). *(Multiply the i-th row with λ)*

iii) $R_{i,j}A$ is the matrix obtained from A by row operation (R3). *(Swap the j-th row with the i-th row)*

Corollary 4.17. *Let $A \in \mathbb{R}^{n\times n}$.*

i) The matrix A is invertible if and only if it is a product of elementary matrices.

ii) If C is an elementary matrix then $\det(CA) = \det(C)\det(A)$.

5 Eigenvalues and eigenvectors

In this section V always denotes a vector space.

Definition 5.1. *Let $F : V \to V$ be a linear map.*

i) A $\lambda \in \mathbb{R}$ is called an **eigenvalue** *of F, if there exist a vector $v \in V$ with $v \neq 0$, such that*

$$F(v) = \lambda v\,. \tag{5.1}$$

ii) A vector $v \in V$ with $v \neq 0$, satisfying (5.1), is called an **eigenvector** *of F with eigenvalue λ.*

Notice that $v = 0$ always satisfies (5.1) for any $\lambda \in \mathbb{R}$, since F is a linear map. This is one of many reasons why $v = 0$ is not called an eigenvector of F.

In the following, we always assume that V is a finitely generated vector space.

Definition 5.2. *Let $F : V \to V$ be a linear map let $\mathrm{id}_V : V \to V$ be the identity map on V.*

i) The polynomial $f_F(\lambda) = \det(F - \lambda\,\mathrm{id}_V)$ is called the **characteristic polynomial** *of F.*

ii) Let $\lambda \in \mathbb{R}$ be an eigenvalue of F. Then the space

$$\begin{aligned} E_\lambda(F) &= \ker(F - \lambda\,\mathrm{id}_V) \\ &= \{v \in V \mid F(v) = \lambda v\} \end{aligned}$$

is called the **eigenspace** *of F with respect to the eigenvalue λ.*

The eigenspace $E_\lambda(F)$ contains therefore all eigenvectors of F with eigenvalue λ and the zero vector.

Definition 5.3. *i) Let* $\dim V = n$. *A linear map* $F : V \to V$ *is called* **diagonalizable** *if there exist a basis* B *of* V*, such that*

$$[F]_B = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

for some $d_1, \dots, d_n \in \mathbb{R}$.

ii) A matrix $A \in \mathbb{R}^{n\times n}$ *is called* **diagonalizable** *if there exists an invertible matrix* $S \in \mathbb{R}^{n\times n}$ *with*

$$S^{-1}AS = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

for some $d_1, \dots, d_n \in \mathbb{R}$.

Lemma 5.4. *Let* B *be a basis of* V *and let* $F : V \to V$ *be a linear map. Then the following two statements are equivalent*

i) The linear map F *is diagonalizable.*

ii) The matrix $[F]_B$ *is diagonalizable.*

Lemma 5.5. *Let* $F : V \to V$ *be a linear map and* $B = (b_1, \dots, b_n)$ *be a basis of* V*, such that all* b_i *are eigenvectors of* F*, i.e.* $F(b_i) = d_i b_i$ *for some* $d_i \in \mathbb{R}$ *and* $i = 1, \dots, n$. *Then* F *is diagonalizable and*

$$[F]_B = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

Conversely, if F *is diagonalizable then there exists a basis of eigenvectors.*

Theorem 5.6. *Let* $v_1, \dots, v_m \in V$ *be eigenvectors of a linear map* $F : V \to V$ *with* <u>*different*</u> *eigenvalues* $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. *Then* $v_1, \dots, v_m$ *are linearly independent.*

Corollary 5.7. *Let* $F : V \to V$ *be a linear map with eigenvalues* $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ *and* $\dim V = n$.

i) If F *has* n *distinct eigenvalues, i.e.* $m = n$*, then* F *is diagonalizable.*

ii) If $B_1, \dots, B_m$ *are bases of* $E_{\lambda_1}(F), \dots, E_{\lambda_m}(F)$*, then* $B_1 \cup \dots \cup B_m$ *are linearly independent.*

iii) The map F *is diagonalizable if and only if*

$$\sum_{j=1}^{m} \dim E_{\lambda_j}(F) = n\,.$$

Definition 5.8. *Let $F : V \to V$ be a linear map and let $\lambda \in \mathbb{R}$ be an eigenvalue of F.*

i) The **algebraic multiplicity** *of λ, denoted by $\mathrm{algmu}_F(\lambda)$, is the multiplicity of λ in the characteristic polynomial f_F.*

ii) The **geometric multiplicity** *of λ is given by $\mathrm{geomu}_F(\lambda) = \dim E_\lambda(F)$.*

Theorem 5.9. *Let $F : V \to V$ be a linear map and $\lambda \in \mathbb{R}$ be an eigenvalue of F. Then*

$$\mathrm{geomu}_F(\lambda) \leq \mathrm{algmu}_F(\lambda)\,.$$

Corollary 5.10. *If F is diagonalizable then $\mathrm{geomu}_F(\lambda) = \mathrm{algmu}_F(\lambda)$ for all eigenvalues λ of F.*

5.1 The spectral theorem

In this section we will just consider the vector space $V = \mathbb{R}^n$. Recall that the **norm** of a vector $x \in \mathbb{R}^n$ is defined by

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}\,, \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n\,.$$

Definition 5.11. *An* **orthogonal map** *is a linear map $F : \mathbb{R}^n \to \mathbb{R}^n$, such that*

$$\|F(x)\| = \|x\|\,, \qquad \forall x \in \mathbb{R}^n\,,$$

i.e. the map F does not change the norm of a vector. We call a matrix $A \in \mathbb{R}^{n\times n}$ orthogonal if $\|Ax\| = \|x\|$ for all $x \in \mathbb{R}^n$.

Recall that the **dot product** $\bullet$ for two vectors $x, y \in \mathbb{R}^n$ is defined by

$$x \bullet y = x^T y = x_1 y_1 + \cdots + x_n y_n \in \mathbb{R}\,.$$

With this the norm of a vector can also be written as $\|x\| = \sqrt{x \bullet x}$.

Lemma 5.12. *For all $x, y \in \mathbb{R}^n$ we have*

$$x \bullet y = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)\,.$$

Proposition 5.13. *A linear map $F : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if*

$$F(x) \bullet F(y) = x \bullet y$$

for all $x, y \in \mathbb{R}^n$.

Recall: We say that x and y are **orthogonal** if $x \bullet y = 0$. A basis $B = (b_1, \ldots, b_n)$ of $\mathbb{R}^n$ is called an **orthonormal basis** if b_i and b_j for $i \neq j$ are orthogonal and $\|b_i\| = 1$ for all i, i.e.

$$b_i \bullet b_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} .$$

Theorem 5.14. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ a linear map and $A = [F]_B$ the matrix of F for $B = (e_1, \ldots, e_n)$. The following statements are equivalent.*

i) F is orthogonal.

ii) A is orthogonal.

iii) For all $x, y \in \mathbb{R}^n$ we have $F(x) \bullet F(y) = x \bullet y$.

iv) A is invertible and $A^{-1} = A^T$.

v) $(F(e_1), \ldots, F(e_n))$ (the columns of A) is an orthonormal basis of $\mathbb{R}^n$.

vi) If $(b_1, \ldots, b_n)$ is an orthonormal basis of $\mathbb{R}^n$ then $(F(b_1), \ldots, F(b_n))$ is also an orthonormal basis.

Corollary 5.15. *i) $A \in \mathbb{R}^{n\times n}$ is orthogonal if and only if A^T is orthogonal.*

ii) If $A, B \in \mathbb{R}^{n\times n}$ are orthogonal then AB is orthogonal.

iii) If B_1 and B_2 are two orthonormal bases, then the change of basis matrix $S_{B_1}^{B_2}$ is orthogonal.

Definition 5.16. *i) An* **eigenbasis** *of a linear map $F : \mathbb{R}^n \to \mathbb{R}^n$ is a basis consisting of eigenvectors of F.*

ii) Let $U \subset \mathbb{R}^n$ be a subspace. A linear map $F : U \to U$ is called **symmetric** *if we have for all $x, y \in U$*

$$x \bullet F(y) = F(x) \bullet y .$$

Theorem 5.17. *(Spectral theorem) Let $U \subset \mathbb{R}^n$ be a subspace and $F : U \to U$ a linear map. Then F is symmetric if and only if there exists an orthonormal eigenbasis of F.*

Corollary 5.18. *A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if and only if there exists an orthogonal matrix $S \in \mathbb{R}^{n\times n}$, such that*

$$S^T A S = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} ,$$

for some $d_1, \ldots, d_n \in \mathbb{R}$.

Lemma 5.19. *Every symmetric linear map $F : U \to U$ has an eigenvalue.*

6 Linear differential equations

Let $x : \mathbb{R} \to \mathbb{R}^n$ be a function written as

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

where the entries $x_1, ..., x_n$ are differentiable functions in $C^{(1)}(\mathbb{R}, \mathbb{R})$. By $x'(t) = \frac{d}{dt}x(t)$ we denote

$$x'(t) = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix}.$$

For a matrix $A \in \mathbb{R}^{n \times n}$ the equation

$$x'(t) = Ax(t)$$

is called a **continuous (linear) dynamical system**.

One dimensional ($n = 1$) continuous dynamical systems have the following solutions:

Proposition 6.1. *Let $a \in \mathbb{R}$. The only solutions to*

$$x'(t) = a\,x(t)$$

in $C^{(1)}(\mathbb{R}, \mathbb{R})$ are given by $x(t) = c\,e^{at}$ for $c \in \mathbb{R}$.

Recall that the space $C^\infty(\mathbb{R}, \mathbb{R})$, the space of **smooth functions**, denotes the space of all functions $f : \mathbb{R} \to \mathbb{R}$ for which derivatives of all orders exist. This means that for any $n \geq 0$ and $f \in C^\infty(\mathbb{R}, \mathbb{R})$, the n-th derivative $f^{(n)} \in C^\infty(\mathbb{R}, \mathbb{R})$ exists. The space $C^\infty(\mathbb{R}, \mathbb{R})$ is a vector space.

Definition 6.2. *i) A* **differential operator of order** *n is a map $T : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ of the form*

$$T(f) = a_0 f + a_1 f' + a_2 f^{(2)} + \cdots + a_n f^{(n)}$$

for some $a_0, a_1, \dots, a_n \in \mathbb{R}$ with $a_n \neq 0$.

(More precisely this is a "linear differential operator of order n with constant coefficients".)

ii) A **linear differential equation** *is an equation of the form $T(f) = g$, where T is a differential operator and $g \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R})$.*

iii) A linear differential equation is called **homogeneous** *if $g = 0$, i.e. if $T(f) = 0$.*

Lemma 6.3. *Let $F : V \to W$ be a linear map between two vector spaces V and W. Assume that $F(v) = w$ for a fixed $v \in V$ and $w \in W$. Then the following two statements are equivalent:*

i) $F(x) = w$.

ii) $x = v + u$ for some $u \in \ker(F)$.

Theorem 6.4. *Let $T : C^\infty(\mathbb{R}, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ be a differential operator of order n. Then we have*

$$\dim(\ker(T)) = n\,.$$

Definition 6.5. *Let $T(f) = a_0 f + a_1 f' + \cdots + a_n f^{(n)}$ be a differential operator of order n. The* **characteristic polynomial** *of T is defined by*

$$p_T(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + \cdots + a_n x^n\,.$$

In the following, T always denotes a differential operator.

Proposition 6.6. *i) The function $e^{\lambda t}$ is an eigenvector of T with eigenvalue $p_T(\lambda)$.*

ii) We have $e^{\lambda t} \in \ker(T)$ if and only if $p_T(\lambda) = 0$.

Corollary 6.7. *i) If $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are distinct, then $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ are linearly independent.*

ii) If p_T has n distinct zeroes $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ then $(e^{\lambda_1 t}, \ldots, e^{\lambda_n t})$ is a basis of $\ker(T)$.

Lemma 6.8. *For two differential operators T_1 and T_2 we have $T_1 \circ T_2 = T_2 \circ T_1$.*

Theorem 6.9. *Let T be a differential operator with characteristic polynomial*

$$p_T(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_r)^{m_r}$$

where $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ with $\lambda_i \neq \lambda_j$ for $i \neq j$.
Then $B = B_1 \cup \cdots \cup B_r$ is a basis of $\ker(T)$, where we have for $1 \leq j \leq r$

$$B_j = (e^{\lambda_j t}, te^{\lambda_j t}, \ldots, t^{m_j - 1} e^{\lambda_j t})\,.$$

Theorem 6.10. *Let T be a differential operator. If $p_T(x)$ contains a factor $((x-a)^2 + b^2)^m$, then*

$$\{e^{at}\cos(bt), e^{at}\sin(bt), te^{at}\cos(bt), te^{at}\sin(bt), \ldots, t^{m-1}e^{at}\cos(bt), t^{m-1}e^{at}\sin(bt)\}$$

are $2m$ linearly independent elements in $\ker(T)$.

Lemma 6.11. *Let $F : U \to V$ and $G : V \to W$ be surjective linear maps between vector spaces U, V, W such that $\ker(F)$ and $\ker(G)$ are finitely generated. Then we have*

$$\dim(\ker(G \circ F)) = \dim(\ker(F)) + \dim(\ker(G))\,.$$