# 5 Eigenvalues and Eigenvectors

Linear Algebra II
G30 Program, Spring 2020

In the previous section we introduced the determinant of a matrix $A$ and, more generally, of a linear map

$$F: V \longrightarrow V.$$

The determinant told us if a linear map is invertible or not. But there are more important properties of a linear map / matrix and in this section we will introduce one of them, which are the eigenvalues and eigenvectors of a linear map / matrix.

General goal: Understand linear maps

$$F: V \longrightarrow V \quad (V \text{ fin. gen.})$$

Strategy: Find subspaces $U \subset V$, such that $F(U) \subset U$

Most important case: $\dim U = 1$, i.e. for $u \in U$

$$F(u) = \lambda \cdot u \quad \text{for some } \lambda \in \mathbb{R}.$$

$\lambda$: Eigenvalue  $u$: Eigenvector.

Calculating Eigenvalues & Eigenvectors has many applications, e.g. Computer vision, machine learning, google pagerank, quantum mechanics (schrödinger eq.) spectroscopy (chemistry), etc.

**Example 17** Consider the lin. map

$$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$x \longmapsto Ax, \quad A = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}.$$

$U = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{u_2} \right\} \subset \mathbb{R}^3$ satisfies $F(u) = u$ for all $u \in U$.

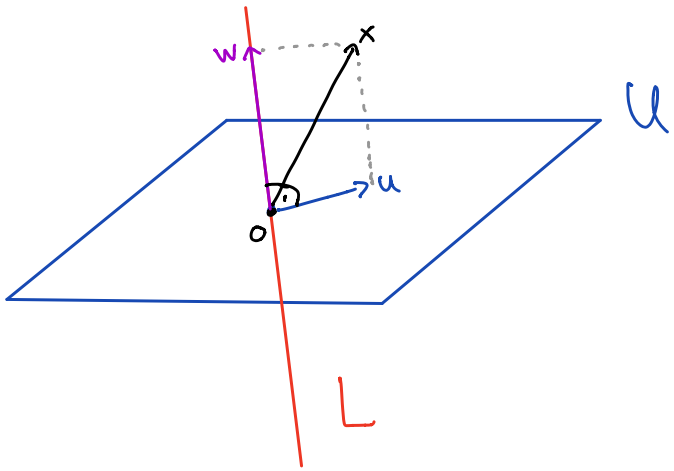
$L = \operatorname{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{u_3} \right\} \subset \mathbb{R}^3$ satisfies $F(w) = -w$ for all $w \in L$.

Moreover $L = U^{\perp}$, i.e. $u_3 \cdot u_1 = u_3 \cdot u_2 = 0$.

and any $x \in \mathbb{R}^3$ can be written uniquely as $x = u + w$ for $u \in U$, $w \in L$.

$F(x) = F(u) + F(w) = u - w$

$\rightsquigarrow$ $F$ is the reflection in the plane $U \subset \mathbb{R}^3$




**Definition 5.1** Let $F: V \to V$ be a linear map.

i) A $\lambda \in \mathbb{R}$ is an <u>eigenvalue</u> of $F$ if there exist a $v \in \mathbb{R}$ with $v \neq 0$, such that

$$F(v) = \lambda \cdot v. \qquad (*)$$

ii) A vector $v \in V$ with $v \neq 0$, satisfying $(*)$, is called an <u>eigenvector</u> of $F$ with eigenvalue $\lambda$.

In Example 17: $\lambda_1 = 1$ is an eigenvalue, since

$$F(u_1) = 1 \cdot u_1. \quad (u_1 \neq 0)$$

$\lambda_2 = -1$ is an eigenvalue, since
$F(u_3) = -u_3 \quad (u_3 \neq 0)$.

In fact any $u \in U$ with $u \neq 0$ is an eigenvector with eigenvalue 1 and any $w \in L$ with $w \neq 0$ is an eigenvector with eigenvalue $-1$.

## How to calculate the eigenvalues & eigenvectors of a linear map $F$

$\lambda$ is eigenvalue of $F$

$\Leftrightarrow \exists v \in V, v \neq 0$ with $F(v) = \lambda v$

$\Leftrightarrow \exists v \in V, v \neq 0 \quad (F - \lambda \cdot id_V)(v) = 0$

($id_V : V \to V$ identity on $V$, i.e. $id_V(v) = v \ \forall v$)

$\Leftrightarrow \exists v \neq 0, \ v \in \ker(F - \lambda id_V)$

$\Leftrightarrow \ker(F - \lambda \cdot id_V) \neq \{0\}$

$\Leftrightarrow \det(F - \lambda \cdot id_V) = 0 \Leftrightarrow \det([F]_B - \lambda \cdot I_n) = 0$

($B$: any basis of $V$, $n = \dim V$)

Thm. 2.5 + Thm. 4.11 $\Rightarrow$ The eigenvalues of $F$ are given as the solutions of $\det(F - \lambda \cdot id_V) = 0$.

The eigenvectors for a $\lambda$ are given as the non-zero elements in $\ker(F - \lambda \cdot id_V)$.

**Example 18** Take again the linear map

$$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$x \longmapsto Ax, \quad A = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}.$$

Eigenvalues:

Want to find the solutions of $\det(A - \lambda \cdot I_3) = 0$

$$\det(A - \lambda \cdot I_3) = \det\left(\begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} \frac{1}{3}-\lambda & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{1}{3}-\lambda & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3}-\lambda \end{pmatrix} \overset{(3)\cdot(3)\cdot(3)}{=} \left(\frac{1}{3}\right)^3 \det\begin{pmatrix} 1-3\lambda & 2 & -2 \\ 2 & 1-3\lambda & 2 \\ -2 & 2 & 1-3\lambda \end{pmatrix}$$

(R1)

We multiplied each row by 3 (R2), i.e the det. changes by $3 \cdot 3 \cdot 3$. So we need to divide by $3^3$.

In general: $A \in \mathbb{R}^{n \times n}$, $\mu \in \mathbb{R}$

$\det(\mu \cdot A) = \mu^n \det(A)$

$$= \left(\frac{1}{3}\right)^3 \det\begin{pmatrix} 3-3\lambda & 0 & -3+3\lambda \\ 0 & 3-3\lambda & 3-3\lambda \\ -2 & 2 & 1-3\lambda \end{pmatrix} = \left(\frac{1}{3}\right)^3 (3-3\lambda)^2 \det\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -2 & 2 & 1-3\lambda \end{pmatrix}$$

(R1)

$$= \left(\frac{1}{3}\right)(1-\lambda)^2 \det\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -3-3\lambda \end{pmatrix}$$

$$= \left(\frac{1}{3}\right)(1-\lambda)^2(-3-3\lambda)\underbrace{\det\begin{pmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}}_{=1} = -(1-\lambda)^2(1+\lambda).$$

$$\Rightarrow \det(A-\lambda\cdot I_3)=0 \iff \lambda=1 \text{ or } \lambda=-1.$$

$\Rightarrow$ The eigenvalues of $F$ are $\lambda_1=1$ and $\lambda_2=-1$ (and there are no more).

Eigenvectors:

$\underline{\lambda_1=1}$: Want to calculate $\mathrm{Ker}(A-\lambda_1 I_3)=\mathrm{Ker}(A-I_3)$:

$$A-I_3=\begin{pmatrix}\frac{1}{3}-1 & \frac{2}{3} & -\frac{2}{3}\\ \frac{2}{3} & \frac{1}{3}-1 & \frac{2}{3}\\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3}-1\end{pmatrix} = \begin{array}{c}\\ -1\\ \\ \end{array}\begin{array}{c}\\ 1\\ \\ \end{array}\begin{pmatrix}-\frac{2}{3} & \frac{2}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & \frac{2}{3}\\ -\frac{2}{3} & \frac{2}{3} & -\frac{2}{3}\end{pmatrix} \sim \begin{array}{c}\\ -\frac{3}{2}\\ \\ \end{array}\begin{pmatrix}-\frac{2}{3} & \frac{2}{3} & \frac{2}{3}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\sim \begin{pmatrix}1 & -1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \Rightarrow \mathrm{Ker}(A-I_3)=\mathrm{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\} \underset{(\text{Ex.17})}{=} U.$$

$\underline{\lambda_2=-1}$: Want to calculate $\mathrm{Ker}(A-\lambda_2 I_3)=\mathrm{Ker}(A+I_3)$

$$A+I_3=\begin{pmatrix}\frac{1}{3}+1 & \frac{2}{3} & -\frac{2}{3}\\ \frac{2}{3} & \frac{1}{3}+1 & \frac{2}{3}\\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3}+1\end{pmatrix} = \begin{array}{c}\frac{3}{2}\\ \frac{3}{2}\\ \frac{3}{2}\end{array}\begin{pmatrix}\frac{4}{3} & \frac{2}{3} & -\frac{2}{3}\\ \frac{2}{3} & \frac{4}{3} & \frac{2}{3}\\ -\frac{2}{3} & \frac{2}{3} & \frac{4}{3}\end{pmatrix} \sim \begin{array}{c}\\ -2\\ 1\end{array}\begin{pmatrix}2 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 2\end{pmatrix}$$

$$-\frac{1}{3}\ 1\begin{pmatrix}0 & -3 & -3\end{pmatrix} \quad -2\begin{pmatrix}0 & 1 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 0 & -1\end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \ker(A + I_3) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\} = L.$$

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$(\dim V = n)$

In general: If $F: V \to V$ is a linear map and $\mathcal{B}$ is any basis of $V$, then $\det(F - \lambda \mathrm{id}) = \det([F - \lambda \mathrm{id}]_{\mathcal{B}})$

$$= \det(\underbrace{[F]_{\mathcal{B}}}_{(a_{ij})} - \lambda \cdot I_n) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}$$

with $b_{ij} = \begin{cases} a_{ij} & i \neq j \\ a_{ii} - \lambda & i = j \end{cases}$

$\Rightarrow \det(F - \lambda \mathrm{id})$ is a polynomial of degree $n$ in $\lambda$.

Eigenvalues = zeros of this polynomial.

Eigenvectors for eisenval. $\lambda$ = non-zero elements in $\ker(F - \lambda \mathrm{id})$.

**Definition 5.2** Let $F: V \to V$ be a linear map.

i) The polynomial $f_F(\lambda) = \det(F - \lambda \mathrm{id}_V)$ is called the characteristic polynomial of $F$.

ii) Let $\lambda \in \mathbb{R}$ be an eigenvalue of $F$. Then the subspace

$$E_\lambda(F) = \ker(F - \lambda \mathrm{id}_V) = \{v \in V \mid F(v) = \lambda v\}$$

is called the eigenspace of $F$ with respect to

the eigenvalue $\lambda$.

- The space $E_\lambda(F)$ therefore contains all eigenvectors of $F$ with eigenvalue $\lambda$ and the zero vector $0$.

- The degree of $f_F(\lambda)$ is $n = \dim V$ and therefore $F$ can have at most $n$ different eigenvalues.

But $F$ can also have no eigenvalues at all.

e.g. $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ has no eigenvalues

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ one eigenvalue

$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ two eigenvalues