Homework 8: Orthogonality & Spectral theorem

Deadline: 5th July, 2020

Exercise 1. (4 Points) Which of the following matrices are orthogonal?

$$A = \frac{1}{5}\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}, \qquad B = \frac{1}{3}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}.$$

Exercise 2. (8 Points) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map with $A = [F]_B$, where $B = (e_1, \dots, e_n)$. Show the following two implications:

i) If $(F(e_1), \dots, F(e_N))$ is an orthonormal basis then A is invertible and $A^{-1} = A^T$.

ii) If $A^{-1} = A^T$ then $F(x) \bullet F(y) = x \bullet y$ for all $x, y \in \mathbb{R}^n$.

Exercise 3. (10 Points) Show that a matrix $A \in \mathbb{R}^{2\times 2}$ is symmetric, i.e. $A = A^T$, if and only if there exists an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

(This is the $n = 2$ case of the spectral theorem (Theorem 5.17), which you are not allowed to use to solve this Exercise. You are also not allowed to use Lemma 5.19 and need to prove it in this special case.)