# Homework 4: Induction & Determinants I

Deadline: 31th May, 2020

**Exercise 1.** (10 Points) Use mathematical induction to prove the following statements.

i) For all $n \geq 1$ we have
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 \,.$$

ii) The number of permutations of $\{1, \ldots, n\}$ is given by $n!$.
(Here $n! = 1 \cdot 2 \cdots n$ denotes the factorial.)

iii) Let $V$ be a vector space which is not finitely generated. Then for any $n \geq 1$ there exist vectors $v_1, \ldots, v_n \in V$ which are linearly independent.

**Exercise 2.** (6 Points) Calculate the determinants of the following matrices. For each matrix, write down all patterns which give a non-zero contribution to the determinant.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 4 & 2 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & 0 & 5 \end{pmatrix}.$$

**Exercise 3.** (4 Bonus points) (Geometric interpretation of the determinant)
We define the vectors $v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, u = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \in \mathbb{R}^2$.

i) Connect the endpoints of the vectors $0$, $v$, $u$ and $v + u$ to get a parallelogram in $\mathbb{R}^2$. (Make a sketch)

ii) Show that the area of this parallelogram is given by $\det \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$, i.e. the determinant of the matrix which has $v$ and $u$ as columns.