Homework 3: Matrix of a linear map

Deadline: 24th May, 2020

The first exercise is a golden week bonus exercise and is just for fun. The solution is almost completely contained in the lecture notes/video but needs to be written out.

Exercise 1. (4 Bonus points) The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2}\,. \qquad (n \geq 2)$$

In this exercise we want to prove the following explicit formula

$$F_n = \frac{1}{2^n\sqrt{5}}\left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right)\,. \tag{0.1}$$

For this follow the following steps:

i) Find a linear map $F : \mathbb{R}^2 \to \mathbb{R}^2$, such that $F^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix}$ for $n \geq 1$, where $F^n = \underbrace{F \circ \cdots \circ F}_{n}$.

ii) We define the following two bases of $\mathbb{R}^2$:

$$B_1 = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)\,, \quad B_2 = \left(\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}\right)\,.$$

Determine the matrices $S^{B_2}_{B_1}$ and $\left(S^{B_2}_{B_1}\right)^{-1}$.

iii) Calculate $[F]_{B_1}$ and $[F]_{B_2}$.

iv) Calculate $[F]^n_{B_1}$ by using

$$[F]_{B_1} = \left(S^{B_2}_{B_1}\right)^{-1} [F]_{B_2}\, S^{B_2}_{B_1}$$

and prove (0.1) by using i).

Exercise 2. (8 Points) We define the matrix $A = \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$.

i) Find vectors $v_1, v_2 \in \mathbb{R}^2$, such that

$$Av_1 = 2v_1\,. \qquad Av_2 = -2v_2\,.$$

ii) Give an explicit formula for A^n when $n \geq 2$ is an even number.
(Hint: Try to find matrices S and D, such that $A = S^{-1}DS$, where D just has entries on the diagonal. Use your result from i) for this).

Exercise 3. (8 Points) We define for $j = 0, 1, 2$ the following Lagrange polynomial functions $L_j \in \mathcal{P}_2$

$$L_j(x) = \prod_{\substack{0 \leq m \leq 2 \\ m \neq j}} \frac{x-m}{j-m}\,.$$

i) Show that $L = (L_0, L_1, L_2)$ is a basis of $\mathcal{P}_2$.

ii) Given a list of data points $(0, y_0), (1, y_1), (2, y_2)$ how can you use the Lagrange polynomials to find a polynomial function $f \in \mathcal{P}_2$, such that $f(j) = y_j$ for $j = 0, 1, 2$?

iii) Let $B = (f_0, f_1, f_2)$ be the basis of $\mathcal{P}_2$, where $f_j(x) = x^j$ for $j = 0, 1, 2$. Calculate the change-of-basis matrix S^L_B.

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