# Homework 1: Vector spaces

Deadline: 3rd May, 2020

**Exercise 1.** (8 Points) Let $X$ be a set, $V$ a vector space, and $\mathcal{F}(X,V)$ the set of all functions from $X$ to $V$. For functions $f,g \in \mathcal{F}(X,V)$ and $\lambda \in \mathbb{R}$, we define the elements $f+g$ and $\lambda f$ in $\mathcal{F}(X,V)$ by

$$(f+g)(x) := f(x)+g(x)\,, \qquad \text{and} \qquad (\lambda f)(x) := \lambda \cdot f(x)\,, \qquad \text{for all } x \in X.$$

Show that $\mathcal{F}(X,V)$ is a vector space with these operations.

**Exercise 2.** (8 Points) Let $V=\mathbb{R}$ and define on $V$ for $u,v \in V$ the new addition

$$u+v := \max(u,v)\,,$$

where $\max(u,v)$ denotes the maximum of $u$ and $v$. For the scalar multiplication we use the usual multiplication of real numbers.
Show that $V$ with these operations is not a vector space. Give a counter example for all properties $(A.1)-(A.4)$ and $(C.1)-(C.4)$, which are not satisfied in [L, Definition 1.1]

**Exercise 3.** (10 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. We saw in the lecture that this is a vector space. Define the following subsets

$$\mathcal{P}_3 = \{f \in \mathcal{P} \mid \deg(f) \leq 3\}\,,$$
$$U = \{f \in \mathcal{P}_3 \mid f(1)=0\} \subset \mathcal{P}_3\,.$$

i) Show that $\mathcal{P}_3$ is a subspace of $\mathcal{P}$.

ii) Show that $U$ is a subspace of $\mathcal{P}_3$.

iii) Determine a basis $B=(b_1,\dots,b_n)$ of $U$.

iv) Determine the coordinate vector $[f]_B$ for the function $f$ given by $f(x)=(x-1)^3$.

v) Extend the basis $B$ to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis $B$ of $U$)

# References

[L] Linear Algebra II - Overview notes, See `https://www.henrikbachmann.com/la2_2020.html`.